Bits, Data Types, and Operations

How Do We Represent Data In A Computer?

At the lowest level, a computer is an electronic machine.

works by controlling the flow of electrons

Easy to recognize two conditions:

1. presence of a voltage – we’ll call this state “1”
2. absence of a voltage – we’ll call this state “0”

Could base state on *value* of voltage, but control and detection circuits would be more complex.

- compare turning on a light switch to measuring or regulating voltage

Computer Is A Binary Digital System

Digital system:

- finite number of symbols

Binary (base two) system:

- has two states: 0 and 1

- Basic unit of information is the *binary digit*, or *bit*.
- Values with more than two states require multiple bits.
  - A collection of two bits has four possible states: 00, 01, 10, 11
  - A collection of three bits has eight possible states: 000, 001, 010, 011, 100, 101, 110, 111
  - *A* collection of n bits has 2^n possible states.

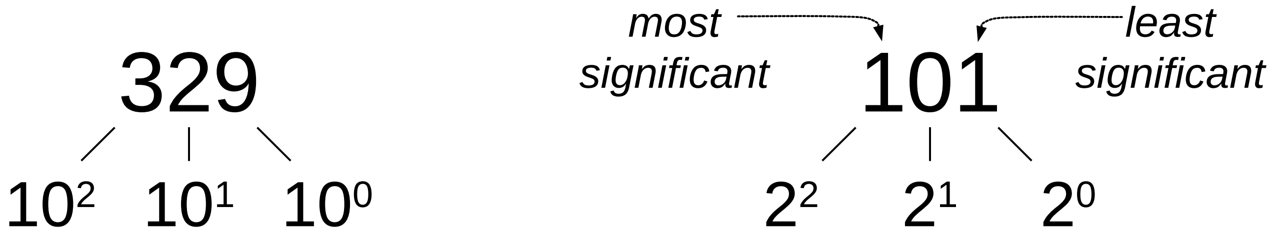
What Kinds of Data Do We Need To Represent?

  - Numbers – signed, unsigned, integers, floating point, complex, rational, irrational, …
  - Text – characters, strings, …
  - Images – pixels, colors, shapes, …
  - Sound
  - Logical – true, false
  - Instructions
  - …
- Data type:
  - *representation* and *operations* within the computer
- We'll start with numbers…

Unsigned Integers

- Non-positional notation
  - could represent a number (“5”) with a string of ones (“11111”)
  - problems?

- Weighted positional notation
  - like decimal numbers: “329”
  - “3” is worth 300, because of its position, while “9” is only worth 9



3x100 + 2x10 + 9x1 = 329

1x4 + 0x2 + 1x1 = 5

Unsigned Integers (cont.)

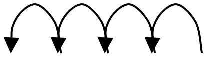
An *n*-bit unsigned integer represents 2^n values:
from 0 to 2^n-1.

2^2	2^l	2^0	
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7

Unsigned Binary Arithmetic

Base-2 addition – just like base-10!

- add from right to left, propagating carry



```
   10010         10010         1111
 + 1001        +  1011     +      1
   11011         11101        10000

                 10111
               +   111
```

Subtraction, multiplication, division,…

Signed Integers

- With n bits, we have 2^n distinct values.
  - assign about half to positive integers (1 through 2^{n-1} -1)
  - and about half to negative (- 2^{n-1} -1 through -1)
  - that leaves two values: one for 0, and one extra
- Positive integers
  - just like unsigned – zero in *most significant* (MS) bit
    00101 = 5
- Negative integers
  - sign-magnitude – set MS bit to show negative,
    other bits are the same as unsigned
    10101 = -5
  - one's complement – flip every bit to represent negative
    11010 = -5
  - in either case, MS bit indicates sign: 0=positive, 1=negative

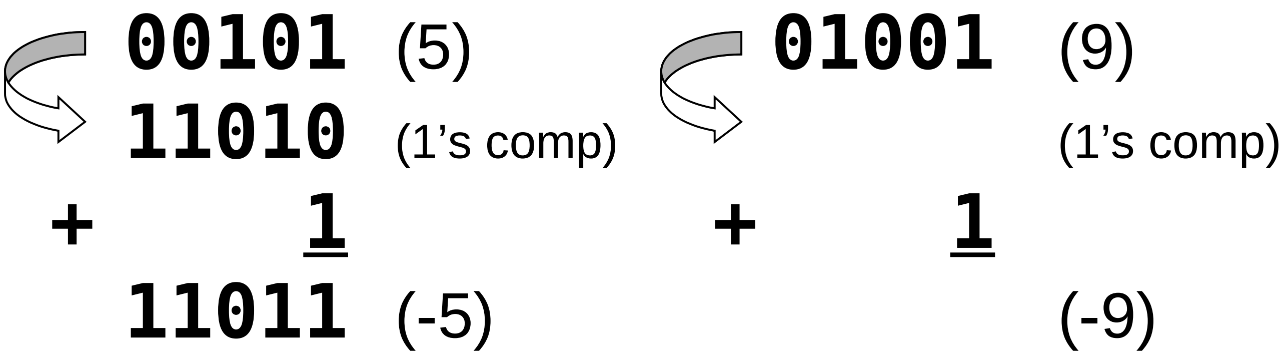
Two’s Complement

- Problems with sign-magnitude and 1’s complement
  - two representations of zero (+0 and –0)
  - arithmetic circuits are complex
    - How to add two sign-magnitude numbers?
      - e.g., try 2 + (-3)
    - How to add to one’s complement numbers?
      - e.g., try 4 + (-3)
- ***Two’s complement*** representation developed to make circuits easy for arithmetic.
  - for each positive number (X), assign value to its negative (-X), such that X + (-X) = 0 with “normal” addition, ignoring carry out

$$\begin{array}{rrl} & \mathbf{00101} & (5) \\ + & \underline{\mathbf{11011}} & (-5) \\ & \mathbf{00000} & (0) \end{array} \qquad\qquad \begin{array}{rrl} & \mathbf{01001} & (9) \\ + & & (-9) \\ & \mathbf{00000} & (0) \end{array}$$

Two’s Complement Representation

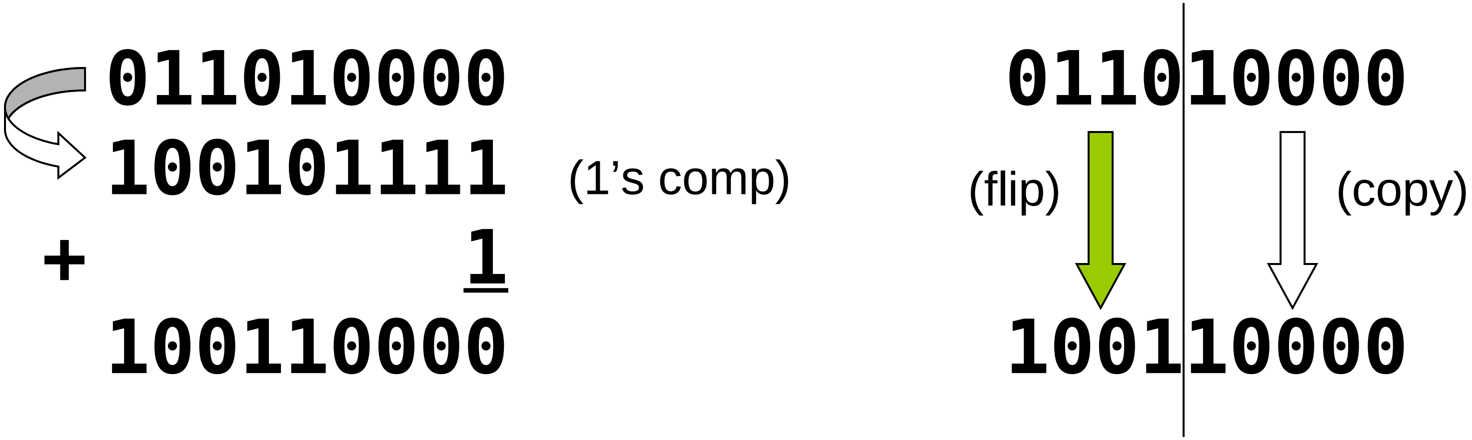
- If number is positive or zero,
  - normal binary representation, zeroes in upper bit(s)
- If number is negative,
  - start with positive number
  - flip every bit (i.e., take the one’s complement)
  - then add one



```
   00101  (5)               01001  (9)
   11010  (1’s comp)               (1’s comp)
+      1                +       1
   11011  (-5)                     (-9)
```

Two’s Complement Shortcut

- To take the two’s complement of a number:
  - copy bits from right to left until (and including) the first “1”
  - flip remaining bits to the left



```
  011010000
  100101111   (1’s comp)
+         1
  100110000
```

```
011|10000
(flip)   (copy)
100|10000
```

Two’s Complement Signed Integers

- MS bit is sign bit – it has weight $–2^{n-1}$.
- Range of an n-bit number: -2^{n-1} through $2^{n-1} – 1$.
  - The most negative number (-2^{n-1}) has no positive counterpart.

-2^3	2^2	2^1	2^0	
0	0	0	0	0
0	0	0	1	1
0	0	1	0	2
0	0	1	1	3
0	1	0	0	4
0	1	0	1	5
0	1	1	0	6
0	1	1	1	7

-2^3	2^2	2^1	2^0	
1	0	0	0	-8
1	0	0	1	-7
1	0	1	0	-6
1	0	1	1	-5
1	1	0	0	-4
1	1	0	1	-3
1	1	1	0	-2
1	1	1	1	-1

Converting Binary (2's C) to Decimal

- If leading bit is one, take two's complement to get a positive number.
- Add powers of 2 that have "1" in the corresponding bit positions.
- If original number was negative, add a minus sign.

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

$$\begin{aligned} X &= 01101000_{\text{two}} \\ &= 2^6+2^5+2^3 = 64+32+8 \\ &= 104_{\text{ten}} \end{aligned}$$

**Assuming 8-bit 2's complement numbers.*

More Examples

$$\begin{aligned} X &= 00100111_{\text{two}} \\ &= 2^5+2^2+2^1+2^0 = 32+4+2+1 \\ &= 39_{\text{ten}} \end{aligned}$$

$$\begin{aligned} X &= 11100110_{\text{two}} \\ -X &= 00011010 \\ &= 2^4+2^3+2^1 = 16+8+2 \\ &= 26_{\text{ten}} \\ X &= -26_{\text{ten}} \end{aligned}$$

**Assuming 8-bit 2’s complement numbers.*

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

Converting Decimal To Binary (2's C)

First Method: ***Division***

1. Find magnitude of decimal number. (Always positive.)
2. Divide by two – remainder is least significant bit.
3. Keep dividing by two until answer is zero, writing remainders from right to left.
4. Prepend a zero as the MS bit; if original number was negative, take two's complement.

$X = 104_{ten}$

104/2 = 52 r0 *bit 0*
52/2 = 26 r0 *bit 1*
26/2 = 13 r0 *bit 2*
13/2 = 6 r1 *bit 3*
6/2 = 3 r0 *bit 4*
3/2 = 1 r1 *bit 5*
1/2 = 0 r1 *bit 6*

$X = 01101000_{two}$

Converting Decimal To Binary (2’s C)

Second Method: ***Subtract Powers of Two***

1. Find magnitude of decimal number.
2. Subtract largest power of two less than or equal to number.
3. Put a one in the corresponding bit position.
4. Keep subtracting until result is zero.
5. Append a zero as MS bit; if original was negative, take two’s complement.

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024

$X = 104_{\text{ten}}$

104 - 64 = 40 *bit 6*

40 - 32 = 8 *bit 5*

8 - 8 = 0 *bit 3*

$X = 01101000_{\text{two}}$

Operations: Arithmetic and Logical

- Recall:
  a data type includes *representation* and *operations*.
- We now have a good representation for signed integers, so let's look at some arithmetic operations:
  - Addition
  - Subtraction
  - Sign Extension
- We'll also look at overflow conditions for addition.
- Multiplication, division, etc., can be built from these basic operations.
- Logical operations are also useful:
  - AND
  - OR
  - NOT

Addition

- As we’ve discussed, 2’s complement addition is just binary addition.
  - assume all integers have the same number of bits
  - ignore carry out
  - for now, assume that sum fits in n-bit 2’s complement representation

```
    01101000 (104)       11110110 (-10)
  + 11110000 (-16)     +          (-9)
    --------
    01011000 (98)                 (-19)
```

**Assuming 8-bit 2’s complement numbers.*

Subtraction

- Negate subtrahend (2nd no.) and add.
  - assume all integers have the same number of bits
  - ignore carry out
  - for now, assume that difference fits in n-bit 2’s complement representation

	01101000	(104)		**11110110**	(-10)
-	**00010000**	(16)	**-**		(-9)
	01101000	(104)		**11110110**	(-10)
+	**11110000**	(-16)	**+**		(9)
	01011000	(88)			(-1)

**Assuming 8-bit 2’s complement numbers.*

Sign Extension

To add two numbers, we must represent them with the same number of bits… but…

If we just pad with zeroes on the left we have a problem:

4-bit	**8-bit**
0100 (4)	**00000100** (still 4)
1100 (-4)	**00001100** (12, not -4)

Instead, replicate the MS bit (sign bit) to fill the bits:

4-bit	**8-bit**
0100 (4)	**00000100** (still 4)
1100 (-4)	**11111100** (still -4)

Overflow

- If operands are too big, then sum cannot be represented as an *n*-bit 2’s complement number.

$$\begin{array}{rrl} & \mathbf{01000} & (8) \\ + & \underline{\mathbf{01001}} & (9) \\ & \mathbf{10001} & (-15) \end{array} \qquad\qquad \begin{array}{rrl} & \mathbf{11000} & (-8) \\ + & \underline{\mathbf{10111}} & (-9) \\ & \mathbf{01111} & (+15) \end{array}$$

We have overflow if:

- signs of both operands are the same, and
- sign of sum is different.

- Another test -- easy for hardware:
  - carry into MS bit does not equal carry out

Logical Operations

- Operations on logical TRUE or FALSE
  - two states -- takes one bit to represent: TRUE=1, FALSE=0

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

A	NOT A
0	1
1	0

- View an n-bit number as a collection of n logical values
  - Operation is applied to each bit independently

Examples of Logical Operations

- AND
  - useful for clearing bits
    - AND with zero = 0
    - AND with one = no change

$$\begin{array}{rr} & \mathtt{11000101} \\ \text{AND} & \underline{\mathtt{00001111}} \\ & \mathtt{00000101} \end{array}$$

- OR
  - useful for setting bits
    - OR with zero = no change
    - OR with one = 1

$$\begin{array}{rr} & \mathtt{11000101} \\ \text{OR} & \underline{\mathtt{00001111}} \\ & \mathtt{11001111} \end{array}$$

- NOT
  - unary operation -- one argument
  - flips every bit

$$\begin{array}{rr} \text{NOT} & \underline{\mathtt{11000101}} \\ & \mathtt{00111010} \end{array}$$

Hexadecimal Notation

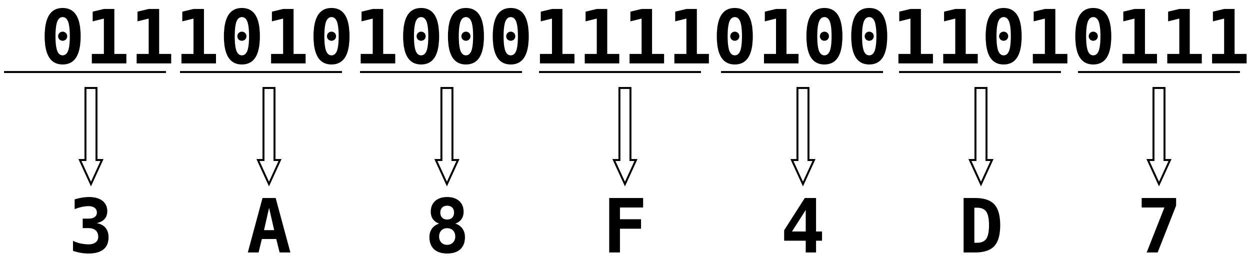
- It is often convenient to write binary (base 2) numbers as hexadecimal (base 16) numbers instead.
  - fewer digits -- four bits per hex digit
  - less error prone -- easy to corrupt long string of 1s and 0s

Binary	Hex	Decimal
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7

Binary	Hex	Decimal
1000	8	8
1001	9	9
1010	A	10
1011	B	11
1100	C	12
1101	D	13
1110	E	14
1111	F	15

Converting From Binary To Hexadecimal

- Every four bits is a single hex digit.
  - start grouping from right-hand side



011101010001111010011010111

3 A 8 F 4 D 7

This is not a new machine representation, just a convenient way to write the number.

Octal Notation

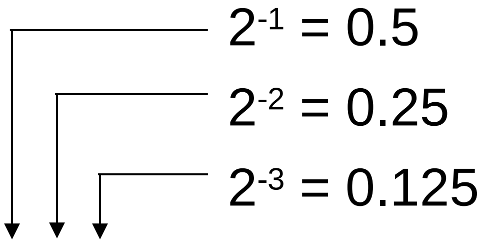
- This is an alternate notation for writing binary (base 2) numbers as octal (base 8) numbers instead.
  - fewer digits -- three bits per octal digit
  - less error prone -- easy to corrupt long string of 1s and 0s

Binary	Octal	Decimal
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7

Binary	Octal	Decimal
1000	10	8
1001	11	9
1010	12	10
1011	13	11
1100	14	12
1101	15	13
1110	16	14
1111	17	15

Fractions: Fixed-Point

- How can we represent fractions?
  - Use a “binary point” to separate positive from negative powers of two -- just like “decimal point.”
  - 2’s comp addition and subtraction still work.
    - if binary points are aligned



$2^{-1} = 0.5$

$2^{-2} = 0.25$

$2^{-3} = 0.125$

```
  00101000.101 (40.625)
+ 11111110.110 (-1.25)
  00100111.011 (39.375)
```

No new operations -- same as integer arithmetic.

Very Large or Small: Floating-Point

- Large values: 6.023 x 10^{23} -- requires 79 bits
- Small values: 6.626 x 10^{-34} -- requires >110 bits

- Use equivalent of “scientific notation”: F x 2^{E}

  Need to represent F (*fraction*), E (*exponent*), and sign.

  IEEE 754 Floating-Point Standard (32-bits):

1b	8b	23b
S	Exponent	Fraction

$$N = (-1)^{S} \times 1.\text{fraction} \times 2^{\text{exponent}-127},\ 1 \le \text{exponent} \le 254$$

$$N = (-1)^{S} \times 0.\text{fraction} \times 2^{-126},\ \text{exponent} = 0$$

Floating-Point Example

Single-precision IEEE floating point number:

1 01111110 10000000000000000000000

sign *exponent* *fraction*

- – Sign is 1 means number is negative.
- – Exponent field is 01111110 = 126 (decimal).
- – Fraction is 0.100000000000… = 0.5 (decimal).

- Value = -1.5 x $2^{(126-127)}$ = -1.5 x 2^{-1} = -0.75.

Text: ASCII Characters

- ASCII: Maps 128 characters to 7-bit code.
  - both printable and non-printable (ESC, DEL, …) characters

00	nul	10	dle	20	sp	30	0	40	@	50	P	60	`	70	p
01	soh	11	dc1	21	!	31	1	41	A	51	Q	61	a	71	q
02	stx	12	dc2	22	"	32	2	42	B	52	R	62	b	72	r
03	etx	13	dc3	23	#	33	3	43	C	53	S	63	c	73	s
04	eot	14	dc4	24	$	34	4	44	D	54	T	64	d	74	t
05	enq	15	nak	25	%	35	5	45	E	55	U	65	e	75	u
06	ack	16	syn	26	&	36	6	46	F	56	V	66	f	76	v
07	bel	17	etb	27	'	37	7	47	G	57	W	67	g	77	w
08	bs	18	can	28	(	38	8	48	H	58	X	68	h	78	x
09	ht	19	em	29	)	39	9	49	I	59	Y	69	i	79	y
0a	nl	1a	sub	2a	*	3a	:	4a	J	5a	Z	6a	j	7a	z
0b	vt	1b	esc	2b	+	3b	;	4b	K	5b	[	6b	k	7b	{
0c	np	1c	fs	2c	,	3c	<	4c	L	5c	\	6c	l	7c	|
0d	cr	1d	gs	2d	-	3d	=	4d	M	5d	]	6d	m	7d	}
0e	so	1e	rs	2e	.	3e	>	4e	N	5e	^	6e	n	7e	~
0f	si	1f	us	2f	/	3f	?	4f	O	5f	_	6f	o	7f	del

Properties of ASCII Characters

- What is relationship between a decimal digit ('0', '1', …) and its ASCII code?
- What is the difference between an upper-case letter ('A', 'B', …) and its lower-case equivalent ('a', 'b', …)?
- Given two ASCII characters, how do we tell which comes first in alphabetical order?
- Are 128 characters enough? (http://www.unicode.org/)

No new operations -- integer arithmetic and logic.

Other Data Types

- Text strings
  - sequence of characters, terminated with NULL (0)
  - typically, no hardware support
- Image
  - array of pixels
    - monochrome: one bit (1/0 = black/white)
    - color: red, green, blue (RGB) components (e.g., 8 bits each)
    - other properties: transparency
  - hardware support:
    - typically none, in general-purpose processors
    - MMX -- multiple 8-bit operations on 32-bit word
- Sound
  - sequence of fixed-point numbers

Other Data Types (continued)

- Some data types are supported directly by the instruction set architecture.

- Other data types are supported by interpreting values as logical, text, fixed-point, etc., in the software that we write.