Markov Process

- A stochastic process is a function whose values are random variables
- The classification of a random process depends on different quantities
  - state space
  - index (time) parameter
  - statistical dependencies among the random variables $X(t)$ for different values of the index parameter $t$. 
Markov Process

◆ State Space
  - the set of possible values (states) that $X(t)$ might take on.
  - if there are finite states => discrete-state process or chain
  - if there is a continuous interval => continuous process

◆ Index (Time) Parameter
  - if the times at which changes may take place are finite or countable, then we say we have a discrete-(time) parameter process.
  - if the changes may occur anywhere within a finite or infinite interval on the time axis, then we say we have a continuous-parameter process.
Markov Process

- In 1907 A.A. Markov published a paper in which he defined and investigated the properties of what are now known as Markov processes.
- A Markov process with a discrete state space is referred to as a Markov Chain.
- A set of random variables forms a Markov chain if the probability that the next state is $S_{(n+1)}$ depends only on the current state $S_{(n)}$, and not on any previous states.
Markov Process

- States must be
  - mutually exclusive
  - collectively exhaustive
- Let $P_i(t) = \text{Probability of being in state } S_i \text{ at time } t$.

$$\sum_{i} P_i(t) = 1$$

- Markov Properties
  - future state prob. depends only on current state
    » independent of time in state
    » path to state
Markov Process

- Assume exponential failure law with failure rate $\lambda$.
- Probability that system failed at $t + \Delta t$, given that it was working at time $t$ is given by

$$1 - e^{-\lambda \Delta t}$$

with

$$e^{-\lambda \Delta t} = 1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \cdots$$

we get

$$1 - e^{-\lambda \Delta t} = 1 - [1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \cdots]$$

$$= \lambda \Delta t - \frac{(-\lambda \Delta t)^2}{2!} - \cdots$$
**Markov Process**

- For small \( \Delta t \)

\[
1 - e^{-\lambda \Delta t} \approx \lambda \Delta t
\]
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- Let $P(\text{transition out of state } i \text{ in } \Delta t) =$
  
  $$\sum_{j \neq i} \lambda_{ij} \Delta t$$

- Mean time to transition (exponential holding times)
  
  $$\frac{1}{\sum_{j \neq i} \lambda_{ij}}$$

- If $\lambda$’s are not functions of time, i.e. if $\lambda_i \neq f(t)$
  - homogeneous Markov Chain
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◆ Accessibility
  - state $S_i$ is accessible from state $S_j$ if there is a sequence of transitions from $S_j$ to $S_i$.

◆ Recurrent State
  - state $S_i$ is called recurrent, if $S_i$ can be returned to from any state which is accessible from $S_i$ in one step, i.e. from all immediate neighbor states.

◆ Non Recurrent
  - if there exists at least one neighbor with no return path.
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- sample chain

Which states are recurrent or non-recurrent?
Markov Process

- Classes of States
  - set of states (recurrent) s.t. any state in the class is reachable from any other state in the class.
  - note: 2 classes must be disjoint, since a common state would imply that states from both classes are accessible to each other.

- Absorbing State
  - a state $S_i$ is absorbing iff

\[
\sum_{j \neq i} \lambda_{ij} \Delta t = 0
\]
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◆ Irreducible Markov Chain
  – a Markov chain is called irreducible, if the entire system is one class
    » => there is no absorbing state
    » => there is no absorbing subgraph, i.e. there is no absorbing subset of states