Competing Risks Analysis of Reliability, Survivability, and Prognostics and Health Management (PHM)

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Abstract—Competing risks analysis is a field of applied statistics with research dating back to the eighteenth century. Starting in the 1980s, the interaction with survival analysis has lead to significant advances in competing risks analysis, especially in dealing with the dependency and identifiability issues, both of which are often intermingled with each other and have been the focus of the controversy surrounding classical competing risks analysis. The usefulness of competing risks analysis in engineering reliability has been recognized since the 1960s, and several important models in competing risks analysis were developed in the context of reliability modeling [e.g., Marshall-Olkin (1967) model]. However, the interaction between competing risks analysis and reliability has gradually withered during the period when significant advances were made in competing risks analysis. Consequently, it seems that the application of competing risks analysis in engineering reliability has fallen behind the theory of competing risks analysis. In particular, the advances in dependence and identifiability research are of extremely important significance in reliability field. We hope that this review article will contribute to the reestablishment of the connections between competing risks analysis and engineering reliability. In perspective, we suggest that the competing risks analysis has great potential in other fields of computer science and engineering, besides engineering reliability. In particular, network reliability and survivability, software reliability and test measurements, prognostics and health management, stand out as fields with very compelling reasons for further exploring.

INDEX TERMS: Competing Risks Analysis, Survival Analysis, Reliability, Prognostic and Health Management, Network Survivability, Software Reliability.

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1. INTRODUCTION.

The term risk analysis, as well as various alternative terms such as risk assessment, management, or evaluation, appears in much of the scientific and engineering literature. However, the underlying quantitative methods may be very different from paper to paper. In this paper, we use the term "*competing risks analysis*" in a well-scoped domain, which originated from population demography, and the foundation of which has been supplemented by the modern survival analysis since the 1980s. According to David and Moeschberger (1978), the history of competing risks analysis can be traced back to the great French mathematician Daniel Bernoulli's research on the risks around the smallpox inoculation back to 1760. What

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Bernoulli addressed was an extremely significant medical and social issue, we still do not have full answer for the question (the issue of dependent failures) he raised.

Despite its long history and its focus on dependency among competing risks, it appears that the field of competing risks analysis has long been battling for its independent identity. Occasionally some statisticians in survival analysis tend to consider competing risks analysis as a field of survival analysis. On the other hand, competing risks analysis community complained, that survival analysis itself was a "lost cause", correctly pointing to the fact that traditional survival analysis is univariate (Crowder 2001). The competing risk analysis community insists that, "if something can fail, it can often fail in one of several ways and sometimes more than one way at a time." (Crowder 2001).

Before the 1980s, the mathematical theory of competing risks analysis was dominantly described with a pair of random vectors. One vector is known as latent failure times, since it is not observable. It can be modeled with a multivariate distribution model, such as multivariate exponential or Weibull distribution. The other vector of random variables is the theoretical lifetimes, as well as the corresponding causes. This is known as classical competing risks analysis. When the dependence between competing risks is introduced, mathematical analysis quickly becomes intractable. The simplification assumptions and methods used by classical competing risks analysis caused controversy and much criticism. Starting around the 1980s, an alternative formulation of competing risks analysis was developed, with the hope to better resolve the issues of failure dependency and distribution identifiability.

Given the close ties between competing risks analysis and survival analysis, we would like to briefly introduce their relationship in the remainder of the introduction. The relationship between survival analysis and competing risks analysis may be confusing. First, we should distinguish survival analysis as univariate and multivariate survival analyses, to avoid causing further confusion when discussing their relationships with competing risks analysis.

As reviewed in several monographs (e.g., Crowder 2001, David and Moeschberger 1978, Hougaard 2000), univariate survival analysis has been dominantly based on the *i.i.d* assumptions (independent and identically distributed) or, at least, on the independent failure assumption. Univariate survival analysis assumes that an individual can only fail once from a single cause; the other failures are simply lump-summed as the censored. Distribution-free regression modeling allows one to investigate the influences of multiple covariates on the failure, and it relaxes the assumption of identical failure distribution, and to some extent the single failure risk restriction. However, the independent failures as well as single failure events are still assumed in the univariate survival analysis. Of course, these deficiencies do not invalidate survival analysis (univariate), and indeed, in many applications those assumptions are realistically valid.

In contrast, competing risks analysis has the tradition of tackling dependent failures. The fundamental difficulty comes from the introduction of dependent failures. This has been demonstrated in both reliability theory and survival analysis. In addition, competing risks analysis is essentially a problem of multivariate systems, and therefore it should be more closely related to multivariate survival analysis (Hougaard 2000). According to Hougaard (2000), multivariate survival analysis generally analyzes three types of survival data: (1) survival times of multiple individuals whose failures can be dependent; (2) repeated occurrences of the same event, known as multiple data; (3) times to several events an individual may experience, known as multiple events. Competing risks analysis addresses a type of multivariate survival data that is indeed different from the three types of data Hougaard (2000) classified, because among the multiple latent or theoretic failure times, only one of them, the minimum, is actually observable. This is because in competing risks analysis, multiple risks compete for the failure of an individual, but the individual can fail only once from only one of the risks. The major challenges of competing risks analysis are resulted from three issues: (1) the dependency between the multiple competing risks, and (2) data censoring, where both failure time and failure cause may be censored. Failure time censoring is straightforward, which means that we do not have complete observation for the failure times of some individuals. The other censoring is the inability to identify the cause of the failure. In engineering reliability, the mask of failure cause is a perfect example of this type of censoring. (3) identifiability issue which will be discussed in section 4. From this short discussion, we can see that competing risks analysis is indeed unique, compared to univariate and multivariate survival analysis. One can expect that it can be as complex as multivariate survival analysis, if not more. This also implies that our expectation to competing risks analysis and even multivariate survival analysis has to be reasonable, at this stage; with Bedford's (2005) words, the validation of competing risks analysis models should be *softer* (i.e., resting on the evaluation of the engineering context).

We would like to point out, we use the term "competing risks analysis" strictly limited to its meanings in survival analysis and competing risks analysis fields such as defined in the two monographs (David and Moeschberger 1978, Crowder 2001). In literature, there are numerous concepts, methods, publications, which use risk analysis and some of which may even use the term competing risks. Since it appears that most of the other risk analyses are often related to the statistical decisions or Bayesian decisions, or other specialized decisions approaches, we do not consider they are relevant to the topics of this paper. In section 2, we review the precise mathematical formulations of the competing risks analysis problems to be discussed. We are not very sure why the competing risks analysis is often prefixed with the "classical" modifier, since we have not seen the counterpart term, the modern competing risks analysis. It might be the case that competing risks analysis is prefixed with classical to distinguish it from the various risks analysis methodologies such as those developed in probabilistic decision theory. Of course, this does not imply that the various risks analysis techniques should be termed the "modern" competing risks analysis. Given that competing risks analysis has been studied since 1760, and significant changes have occurred since the 1980s, due to the extensive interactions with survival analysis, we suggest referring to the latter competing risks analysis as *modern* competing risks analysis, *purely* for convenience of reference, if this has not been suggested already.

Classical competing risks analysis has been the major mathematical tool for actuarial and population demographic studies. *Modern* competing risks analysis has been dominantly applied in biomedical and public health research. The classical competing risks analysis indeed was applied to reliability modeling and a few important competing risks models were developed in the context of reliability theory. However, it appears that the links between *modern* competing risks analysis and reliability modeling have not been fully established. Undoubtedly, modern competing risks analysis should be an important tool for engineering reliability modeling.

This article is the second in a four-part series in which we review state-of-the-art research in survival analysis, competing risks analysis, and multivariate survival analysis as well as their applications to engineering reliability and computer science. The other three articles discuss univariate (Ma and Krings 2008a) and multivariate survival analysis (shared frailty and multi-state modeling) (Ma and Krings 2008b & c) respectively. The paper has two primary objectives: (1) to briefly introduce essential models and methodologies of competing risks analysis and review its applications in reliability analysis and IEEE related engineering fields. (2) to present our suggestions and opinions on the potential applications of competing risks analysis to broader computer science and engineering fields, such as the reliability and survivability of computer networks, software reliability and test measurements, and prognostics and health management (PHM). Due to the page limitation, we can only focus on the most important aspects, which in our opinion are the formulations of major competing risks analysis models and underlying failure or dependence mechanisms. For comprehensive and detailed treatments such as statistical inferences of these models, one should refer to the excellent monographs by David and Moeschberger (1978), Crowder (2001) and Pintilie (2006) as well as Bedford's (2005) recent survey paper.

2. General Formulations of Competing Risks Analysis.

There are two general notation systems for the formulations of the competing risks analysis models, reviewed by David and Moeschberger (1978) and Crowder (2001), respectively. To facilitate further discussion, we briefly discuss both notation systems in this section. In addition, we made minor symbol adjustment to keep consistent with the general survival analysis field, and also to be consistent with our three other articles in the series on survival analysis and multivariate survival analysis (Ma and Krings 2008a,b&c).

2.1. Competing Risks Analysis Formulation Reviewed by David and Moeschberger (1978).

This section is mainly drawn from David and Moeschberger (1978). Let C_l ($l=1,2,...,k$) be the k competing risks or causes of failure for individuals in a population. Generally, the term risk should be used before the failure occurs, and the cause is more appropriate afterwards. These k risks compete for the failures of the individuals, but an individual may fail from one of the k causes only and fail only once. Therefore, two sets of random variables are adopted: one set is theoretical and generally not observable, and the other is actually observable.

Let random variable Y_i ($i=1,2,\ldots,k$) represent an individual's lifetime with the assumption that the particular risk C_i were the only risk present. Denote the cumulative distribution function ($c.d.f$) of Y_i by $F_i(x) = \Pr\{Y_i \le x\}$ and corresponding probability density function ($p.d.f.$) as $f_i(x)= p_i(x)$. If not all other k-1 risks excepting C_i can be excluded, Y_i may not be observed. What may be observed is the minimum Z of the k theoretical lifetimes, and the corresponding cause of Z. That is, $Z = \min(Y_1, Y_2, ..., Y_k)$. If Z exceeds x, then all of the Y_i must also exceed x,

$$Pr\{Z>x\} = Pr\{Y_1>x, Y_2>x, ..., Y_k>x\} \tag{1}$$

Obviously, $Pr\{Z > x\}$ is the counterpart of the survival function in univariate survival analysis, and it is denoted as:

$$S_z(x) = Pr\{Z > x\} = 1 - F_z(x)$$

Further, the conditional failure rate function for Z is defined as $r_z(x)$,

$$r_z(x) = f_z(x)/S_z(x) \tag{2}$$

Various terms for $r_z(x)$ are adopted in different application fields, such as: hazard rate, force of mortality, force of decrement, age-specific death (failure) rate, intensity function, etc.

Now let $g_i(x)dx$ (i =1, 2, ..., k) be the probability of failure from cause C_i in $(x, x+dx)$, in the presence of all k risks. Assume that the probability of more than one failure in dx is negligible [order of $(dx)^2$], then $r_z(x)$ is the probability of failure in dx from any cause, conditional on the survival to time x.

$$r_z(x) = \sum_{i=1}^{k} g_i(x) \quad (3)$$

This indicates that the total force of mortality is the sum of the component forces. Up to this point, the risks C_i are not required to act independently.

When introducing the independence assumption, equation (1) becomes:

$Pr\{Z>x\} = Pr\{Y_1>x, Y_2>x, ..., Y_k>x\}$
$= Pr\{Y_1>x\} Pr\{Y_2>x\} ... Pr\{Y_k>x\}$

or

$$S_z(x) = \prod_{i=1}^{k} S_i(x) \quad (4)$$

With the independent assumption,

$$g_i(x) = p_i(x) \prod_{j=1, j\neq l}^{k} S_j(x) / S_z(x) = p_i(x) / S_i(x)$$

This is the failure rate function for Y_i, and called cause-specific failure rate or marginal intensity function. Therefore, for independent risks, there is:

$$g_i(x) = r_i(x) \qquad i = 1, 2, ..., k \quad (5)$$

This equation implies that the probability of failure from C_i in dt, conditional on the survival to t, is not influenced by the simultaneous existence of other k-1 risks. From (3) and (5), the following relationships can be derived.

$$r_z(x) = \sum_{i=1}^{k} r_i(x) \quad (6)$$

$$r_z(x) = -\frac{d}{dx} \ln S_z(x) \quad (6b)$$

$$r_i(x) = -\frac{d}{dx} \ln S_i(x) \quad (7)$$

Besides the above definitions, which are common with standard reliability terminologies, there are three additional definitions, which are widely used in demography but not often in reliability theory. *Crude Probability of Failure* measures the failure from a specific cause in the presence of other risks. *Net Probability of Failure* measures the hypothetical probability of failure if specific risk is the only risk present. *Partial Crude Probability* of failure is the probability of failure from a specific risk in the presence of all risks except that one or more risks are eliminated. These terms can be defined by $r_i(x)$, $r_z(x)$, and $g_i(x)$ for a time interval (a, b) (Chiang 1968, 1984, David and Moeschberger 1978). The following are the definitions for the terms just mentioned. Chiang (1991) in a review on the competing risks analysis in public health further discussed these definitions and their computation, as well their extensions.

David and Moeschberger (1978) derived the formula for the three statistics. The *net failure probability* from cause C_i in the interval *(a, b)*, $q_i(a, b)$ is:

$$q_i(a,b) = 1 - \exp\left[-\int_a^b r_i(x)dx\right] \quad (8)$$

The *crude failure probability* from cause C_i, in the presence of all causes, $Q_i(a, b)$, is expressed as:

$$Q_i(a,b) = \int_a^b g_i(x) \exp\left[-\int_a^x r_z(t)dt\right]dx \quad (9)$$

The *partial crude failure probability* with C_j eliminated is:

$$Q_{ij}(a,b) = \int_a^b g_i^{-j}(x) \exp\left[-\int_a^x r_z^{-j}(t)dt\right]dx \quad (10)$$

where $g_i^{-j}(x)dx$ and $r_z^{-j}(x)$ are the failure probability in dt from cause C_i and the hazard rate, in the absence of cause C_j, respectively.

Equations (8)–(10) are not restricted by the assumption of independent risks. When the risks are independent, they can be simplified by replacing $g_i(x)$ with $r_i(x)$.

This is a convenient place to explain one of the earliest studies of competing risk analysis, Daniel Bernoulli's research on smallpox inoculation, which was mentioned in the introduction. As reviewed by David and Moeschberger (1978) who cited Karn (1931)'s introduction on Daniel Bernoulli's research on the risks around the smallpox inoculation back to 1760. This account is based on David and Moeschberger (1978) review. The question Bernoulli asked was: if the smallpox could be removed in a given population, what would be the effect on the population mortalities at different ages? With the terms introduced by David and Moeschberger (1978), as briefly introduced above, Bernoulli's question would be equivalent to the computation of the *partial crude failure probability*, assuming that smallpox was the risk eliminated and all other risks were lumped together. Bernoulli indicated that his approach rested on a crucial assumption—that individuals immune to smallpox were of the same susceptibility to other risks as the rest of the population. This implies that the risks were independent, and the assumption would not hold if there exists individual variation in the vulnerability to smallpox. The dependency among failures is an issue not yet resolved fully, despite the significant advances in recent years as reviewed in Moeschberger and Klein (1995), Rotnitzky et al. (2007), Chen et al. (2007).

2.2. Competing Risks Analysis Formulation Reviewed by Crowder (2001).

This section introduces the formulation reviewed by Crowder (2001) and the contents are mainly drawn from this reference. We will have a comparison of this formulation with that of David and Moeschberger (1978) later in the section. Again, we adopt minor nonessential adaptations to the notations, to try being consistent in the whole article.

There are two data sets, the time to failure (or survival time) T, and the failure cause (or mode, type) C. T is a continuous random variable with positive value, but C is a set of a handful of labeled discrete values, such as positive integers, 1, 2, ... k.

The key difference between Crowder (2001) and David & Moeschberger (1978) seems that Crowder treats the competing risks framework as a bivariate distribution of T and C. There is one and only one cause in the set of k causes for each failure observed. The elements of C are called risks before failures and causes afterwards. The risks compete for the cause.

In reliability analysis, C might be faulty components in a series system and T is the breakdown time of the system; or C might be the failure modes, such as asymmetric vs. symmetric, etc.

The *sub-distribution function* is defined as:

$$F(j,t) = \Pr(C = j, T \le t) \tag{11}$$

Equivalently, the *sub-survivor function* is defined as:

$$S(j,t) = \Pr(C = j, T > t) \tag{12}$$

Obviously, $S(j,t) + F(j,t) = p_j$

where $p_j = \Pr(C = j) = F(j,\infty) = S(j,0)$

represents the marginal distribution of C.

$F(t, j)$ does not form a proper distribution function because it only accumulates to the value p_j rather than 1 at $t = \infty$. Implicitly, $\sum p_j = 1$, and $p_j > 0$ is assumed. Similarly, $S(j, t)$ is not a proper survivor function because it is not the probability that $T>t$ for failure type j; that probability is a conditional probability, $Pr(T>t|C=j) = S(j,t)/p_j$.

The sub-density function $f(j, t)$ for continuous T is represented by $-dS(j,t)/dt$.

The marginal survivor function and marginal density of T can be calculated with:

$$S(t) = \sum_{j=1}^{k} S(j,t) \tag{13}$$

$$f(t) = -dS(t)/dt = \sum_{j=1}^{k} f(j,t) \tag{14}$$

The related conditional probabilities are:

$$Pr\,(\text{time } t \mid \text{cause } j) = f(j, t) / p_j \tag{15}$$

$$Pr\,(\text{cause } j \mid \text{time } t) = f(j, t) / f(t) \tag{16}$$

$Pr(\text{time } t \mid \text{cause } j)$ represents distribution of failure time caused by j, for example, the survival time distribution of cancer patients in a biomedical experiment. Similarly,
$Pr(\text{cause } j \mid \text{time } t)$ gives distribution of risks faced by a specific *age group*, for example, the failure modes of a system at particular time.

One still needs to define hazards related functions: the *sub-hazard function* and *overall hazard function*. The overall hazard rate from all causes is defined as:

$$\begin{aligned} h(t) &= \lim_{\delta\to 0} \Pr(T \le t+\delta) \mid T > t) / \delta \\ &= f(t)/S(t) = -d\log[S(t)]/dt \end{aligned} \tag{17}$$

The *sub-hazard* function is defined as:

$$\begin{aligned} h(t,j) &= \lim_{\delta\to 0} \Pr(C = j, T \le t+\delta) \mid T > t) / \delta \\ &= f(t,j)/S(t) \end{aligned} \tag{18}$$

Similar to the arguments in equation (3), $h(t) = \sum_{j=1}^{k} h(t,j)$.

where $h(t, j)$ is the cause-specific hazard function, or the marginal hazard function in the latent failure times setup.

Both Crowder (2001) and David and Moeschberger (1978) are essentially equivalent. One may easily map them with each other. For example, equation (11) and (12) are the counterparts of distributions and survivor functions in David and Moeschberger (1978), except Crowder combines causes and failure times into a bivariate probability model. Equation (13) is equivalent to (3), both of which do not assume independent risks; the additivity is due to what we call the *exclusive* assumption, rather than independent assumption. The *exclusive* assumption implies that the probability of more than one failure in dx is negligible, which is often realistic. Equation (14) is simply the p.d.f. expression of the *exclusive* assumption.

David and Moeschberger's (1978) equations (8)-(10), although they appear very different from Crowder's (2001) equation (15)-(16), fulfill similar functions. It seems to us that Crowder's (2001) notation is more powerful and elegant in expressing high level framework, but David and Moeschberger's (1978) notation seems to be more convenient for analysis and also more consistent with other related fields such as reliability analysis.

2.3. Censoring in Competing Risks Analysis

In competing risks analysis, censoring can be simply treated as one of the competing risks, although in recent years, the survival analysis approach is often used to address censoring. However, treating competing risks as censoring is a very tempting simplification, especially for the sake of evaluating the major risk in concern, this naive simplification will cause bias and is not the proper way to analyze survival data (Putter et al. 2007).

Of course, censoring is not limited to the failure times *T,* and the censoring of failure causes can occur in many studies. For example, after a failure, the exact cause of failure may be unidentifiable and often is narrowed down to a certain range of causes. Interestingly, if the *C* (the set of causes) is not observed, competing risks analysis becomes standard survival analysis (Crowder 2001).

3. PARAMETRIC APPROACH (DISTRIBUTIONS-SPECIFIC APPROACH)

Similar to univariate survival analysis, there are generally two categories of approaches to study competing risks analysis. With the category discussed in this section, it is assumed that the probability distribution form of the underlying lifetime or failure time process is known, e.g., multivariate exponential or Weibull distributions, but the parameters of the distribution needs to be estimated. Maximum likelihood estimation, or more recently, Bayesian estimation, are often used to determine the values of the parameters. Once the distribution model and parameters are determined from observation data, the various competing risks analysis problems are turned into standard statistical exercises, such as comparing survivor or reliability functions, computing mean, median and various survivor quantiles (e.g., 50% quantile would be the median lifetime). What is special with competing risk analysis is that the results are expressed as conditional probabilities [e.g., equation (15), (16)]. The second category of approach does not require the exact distribution form to be known and will be discussed in section 6. In each category, we separate discussions into two types: the independent risks *versus* dependent risks. The latter type is far more difficult and many of the problems are analytically intractable.

The joint distributions or survival functions for theoretic survival times in competing risks analysis are multivariate. However, the traditional multivariate statistical analysis is hardly applicable to competing risks analysis, because the multivariate analysis has been developed based on multivariate normal distributions. One may easily conjecture that distributions, such as multivariate exponential and multivariate Weibull distribution, should be the most likely distributions in competing risks analysis.

Pintilie (2006) summarized major parametric models formulated with latent failure times approach (adopted by David and Moeschberger 1978) in a table. The models described include exponential independent, exponential dependent, Weibull, and Marshall-Olkin (1967) model.

3.1. Parametric Competing Risk Analysis with Independent Risks.

David and Moeschberger (1978) reviewed the general maximum likelihood functions for estimating distribution parameters in the competing risks analysis. Even if the independent risks are assumed, there is still significant complexity with competing risks analysis, due to the various censoring involved in collecting the experiment data. They summarized five different scenarios: (i) All lifetimes and associated causes of failures are observed, the simplest case but often least realistic. (ii) Lifetimes are censored. (iii) Lifetimes are grouped into intervals. (iv) Possible immunity exists in some individuals; this is related to the intuitive phenomenon that when one introduces additional risks other than C_i the failure probability from cause C_i is non-increasing. (v) The combinations of (i)-(iv). The complexity of likelihood function generally increases with more complex data structure. When observation censoring is involved, the partial likelihood function used in survival analysis is used.

Here, we only list the results of likelihood function for the case(i), all lifetimes and associated failure causes are known. The following discussion is based on David and Moeschberger (1978). Using the notation in equation (1), Y_i is the theoretic lifetime when C_i is the only cause of failure (i = 1, 2, ..., k). The observed time X_i is conditional on knowing the cause C_i. In the presence of all risks, only the smallest $Z = \min(Y_j)$ is in fact observable. With David and Moeschberger's (1978) notations, this can be written as:

$$X_i = Y_i \mid Y_i = \min_j (Y_j) \tag{19}$$

Let the probability of failure due to cause C_i be

$$\pi_i = \Pr\{Y_i = \min_j Y_j\} \quad \pi_j > 0, \quad \sum_{j=1}^{k} \pi_j = 1 \tag{20}$$

From (19) plus the independence assumption of the Y_i, the *p.d.f.* of X_i is, ignoring the lower order items,

$$f_i(x) = \frac{1}{\pi_i} p_i(x) \prod_{j=1, j\neq i}^{k} S_j(x) \tag{21}$$

Now, if there are N_i individuals that fail from cause C_i and X_{ij} denotes the lifetime of the j-th individual failing from the cause C_i (i =1, 2, ..., k; j=1, 2, ... n_i), conditional on $N_i = n_i$ (i = 1, 2, ..., k), the joint p.d.f. of the X_{ij} would be:

$$\begin{aligned} &f(x_{11},...,x_{1n_1},...,x_{k1},...,x_{kn_k}) \\ &= \prod_{i=1}^{k} \frac{1}{\pi_i^{\,n_i}} \prod_{j=1}^{n_i} p_i(x_{ij}) \prod_{l=1, l\neq i}^{k} S_l(x_{ij}) \end{aligned} \tag{22}$$

Assuming N_i are random variables with the multinomial probability function:

$$\begin{aligned} &f(n_1,...,n_k) = \frac{n!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} \pi_i^{\,n_i} \\ &n = \sum_{i=1}^{k} n_i \end{aligned} \tag{23}$$

Hence the likelihood function of interest is:

$$L = \frac{n!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} \prod_{j=1}^{n_i} p_i(x_{ij}) \prod_{l=1, l \neq i}^{k} S_l(x_{ij}) \quad (24)$$

The (24) can be rearranged as:

$$L = \frac{n!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} L_i \quad (25)$$

where

$$L_i = \left(\prod_{j=1}^{n_i} p_i(x_{ij}) \right) \left(\prod_{l=1, l \neq i}^{k} \prod_{j=1}^{n_l} S(x_{lj}) \right) \quad (26)$$

Equations (25) and (26) show that if each risk follows a different theoretical distribution, the parameter estimation for each distribution can be performed individually by maximizing each L_i.

In this paper, we generally do not discuss detailed parameter estimations, such as the derivations of likelihood functions. However, its importance cannot be overemphasized since it is the parameter estimation under censoring that makes survival analysis or competing risks analysis unique. The processing of information censoring depends on both types of censoring (left, right, random, type-I, type-II, etc) as well as the distribution forms of the censored observations. A significant portion of research efforts in the survival analysis community has been spent on the studies of censoring. In some occasions, researchers from other fields even tried to improve the estimation procedures by introducing some general-purpose optimization techniques, notably, artificial neural networks (ANN). There are two potential pitfalls with the superimposing of black-box ANN on native survival analysis methods: (1) Survival analysis and competing risks analysis are well tuned to maximally extract the partial information in censored observations, and it may be difficult for black-box approaches to outperform the native methods. Several so-called neural survival analysis procedures failed to produce significant improvements or improvements at all (Ma and Krings 2008a). In this aspect, introducing additional optimization may be an unnecessary complication. (2) In some of the artificial neural survival analysis studies, the resulted procedures lost the capability to process censoring; obviously, this type of extensions should be avoided.

3.1.1. <u>Exponential Lifetime Distributions</u>

Exponential distribution is unique in the notion that it is the only continuous distribution with a constant hazard rate and lacking of memory property In addition, its relation with the Poisson process gives it justification for some failure processes (David and Moeschberger's 1978). Despite its limitation in adequately describing failures in practice, exponential distribution is always important as a baseline for studying the departures from constant hazards.

The *p.d.f.* of exponential distribution is:

$$p_i(y) = \lambda_i \exp(-\lambda y) \qquad \lambda > 0,\ y > 0,\ i = 1,2,...,k \quad (27)$$

Substitute $p_i(y)$ into equation (26), the likelihood component of interests is:

$$L_i = \left(\frac{1}{\lambda_i} \right)^{-n_i} \exp\left[-\lambda_i \left(\sum_{l=1}^{k} \sum_{j=1}^{n_l} x_{lj} \right) \right] \quad (28)$$

The maximum likelihood estimate is:

$$\lambda_i = n_i / \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} \quad (29)$$

It can be derived based on (21) (David and Moeschberger 1978) and the fact $\pi_i = \lambda_i / \lambda$ that

$$f_i(x) = \lambda \exp(-\lambda x) \qquad x > 0 \quad (30)$$

where $\lambda = \sum_{i=1}^{k} \lambda_i$

Therefore, the observed lifetimes are identically distributed regardless of the failure causes. Furthermore, the failure rate observed is the sum of the failure rates of all competing risks. Also the *p.d.f* of $Z = \min_l \{Y_l\}$ is identically equal to that of the X_i in (30).

3.1.2. <u>Weibull Lifetime Distributions</u>.

With two-parameter Weibull distribution, the *p.d.f.* of *y* is:

$$p_i(y) = (\lambda_i \beta_i y_i^{\beta_i - 1}) \exp[-\lambda_i y^{\beta_i}] \quad y > 0, \beta_i > 0, \lambda_i > 0 \quad (31)$$

Similar to the case of exponential distribution, when equation (31) is substituted into (26), the maximum likelihood estimations of the parameters β_i, λ_i can be derived, except that the explicit solutions for the likelihood functions are not available and numerical computation is used to estimate the parameters.

If there is no censoring, the *p.d.f.* for observed failure time from cause C_i can be derived based on equation (21), upon noting that $\pi_i = \lambda_i / \lambda$, is:

$$f_i(x) = (\lambda \beta x^{\beta - 1}) \exp[-\lambda x^{\beta}] \qquad x > 0 \quad (32)$$

Thus, the observed lifetimes are again identically distributed regardless of the failure causes. Also the *p.d.f* of $Z = \min_l \{Y_l\}$ is identically equal to that of the X_i in (32).

It should be pointed out that when censoring is involved the likelihood functions are much more complex. Especially when random censoring occurs, the distribution of censoring times further complicates the likelihood function. For the detailed derivation of the above models, readers are referred to David and Moeschberger (1978) where this section is based on.

3.2. Parametric Competing Risk Analysis with Dependent Risks.

Independence of risks has dominated much of the competing risk analysis and univariate survival analysis, similar to the situation in the reliability theory. As shown by David and Moeschberger (1978), one way researchers tried to deal with dependence is by selectively grouping the distinguishable risks into k categories, labeled C_i (i = 1, 2, ... k); within each group, the risks are similar and possibly dependent, but between groups, hopefully there is no dependencies. For example, in medical studies, it is reasonable to assume deaths from cardiovascular diseases and accidents or violent crimes are independent.

Assume the theoretical failure times Y_l ($l = 1, 2, ..., k$) have a continuous joint distribution with *p.d.f.* $p(y_1, y_2, ..., y_k)$. We repeat equations (19) and (20), for the observed lifetime X_i ($i = 1, 2, ..., k$) conditional on the knowing failure cause C_i, and the probability (π_i) of failure due to cause C_i,

$$X_i = Y_i \mid Y_i = \min_j(Y_j)$$

$$\pi_i = \Pr\{Y_i = \min_j Y_j\} \qquad \pi_j > 0, \qquad \sum_{j=1}^{k} \pi_j = 1 \tag{33}$$

The *p.d.f.* of X_i is:

$$f_i(x) = \frac{1}{\pi}\int_x^{\infty} ... \int_x^{\infty} p(y_1,..., y_{i-1}, x, y_{i+1},...y_k) \prod_{l=1, l\neq i}^{k} dy_l$$

$$= \frac{1}{\pi} p_i(x) \int_x^{\infty} ... \int_x^{\infty} p(y_1,..., y_{i-1}, x, y_{i+1},...y_k \mid Y_i = x) \prod_{l=1, l\neq i}^{k} dy_l \tag{34}$$

In the case of independence, (34) reduces to (21).

The likelihood functions with censoring for (34) were given by David and Moeschberger (1978). However, even with numerical computation, the parameters estimation from the derived likelihood functions is very challenging, since the derived likelihood functions are extremely complex.

Bivariate and multivariate exponential distributions receive significant attention in literature. Even for bivariate exponential distribution, the resulting model seems still complex enough to deter wide applications in practical data analysis. Beyond exponential distribution, Weibull distribution might be the only model studied in dependent competing risks analysis. In this section, we show examples discussed in David and Moeschberger (1978) first and then give a brief review of major studies not reviewed in David and Moeschberger (1978) and Crowder (2001) in the next section. The repetition of David and Moeschberger (1978) here helps us illustrate the major issues involved in the dependent competing risks analysis with parametric modeling; therefore, it may simplify the review work in late part of this article.

Marshall and Olkin (1967) presented the idea of obtaining other multivariate distributions by transforming variables. This is a powerful technique to expand the results from multivariate exponential distribution and has been used by others (David and Moeschberger 1978). Marshall and Olkin's (1967) bivariate exponential distribution has the survival function:

$$\begin{aligned} S(y_1, y_2) &\equiv \Pr\{Y_1 > y_1, Y_2 > y_2\} \\ &= \exp[-\lambda_1 y_1 - \lambda_2 y_2 - \lambda_{12}\max(y_1, y_2)] \end{aligned} \tag{35}$$

Marshall and Olkin (1967) derived the model from shock damage model described with Poisson process; this is therefore a competing risks analysis model originated from reliability field. There are three independent Poisson processes, $Z_1(t, \lambda_1)$, $Z_2(t, \lambda_2)$, and $Z_{12}(t, \lambda_{12})$, the number of fatal shocks in time t acting on component 1, component 2 and both 1 and 2 simultaneously, respectively.

Let U_1, U_2, U_{12} be the exponentially distributed times to the first arrival of events in the corresponding Poisson processes, then Y_1=min (U_1, U_{12}), and Y_2=min (U_2, U_{12}) and

$$\Pr(Y_1 = Y_2) = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}} \tag{36}$$

The Marshall and Olkin (1967) model can describe a system made of two series components, and any of the three kinds of fatal shocks leads to system failure. When the system actually fails, one can identify U_1, U_2, or U_{12}, which are individual's failure time from cause 1, 2, or both causes, respectively.

This example shows one useful approach to address dependent risks. As demonstrated by David and Moeschberger (1978), when the dependence between two risks is purely owing to the chance of simultaneous failure from both risks, one may artificially introduce the third risk corresponding to simultaneous failure and then treat the three risks as independent.

Similar to Marshall and Olkin (1967), by taking the transformation

$$Y_1' = Y_1^{1/\beta_1}, \quad Y_2' = Y_2^{1/\beta_2} \tag{37}$$

Moeschberger (1974) derived the bivariate Weibull distribution.

Equation (35) can be extended to k-dimensional exponential distribution by:

$$S(y_1, y_2,..., y_k) = \exp\{-\sum_{i=1}^{k} \lambda_i y_i - \sum_{i<j} \lambda_{ij} \max(y_i, y_j) - \sum_{i<j<l} \lambda_{ijl} \max(y_i, y_j, y_l) - ... - \lambda_{1...k} \max(y_1,..., y_k)\} \tag{38}$$

Given the complexity of the general form, it is understandable that most of the dependent competing risks analysis is ad-hoc, parsimoniously adding necessary complexity.

When the assumption of independent risks is violated, it is often only realistic to determine the bounds of marginal survival function. Klein and Moeschberger (1988) presented such an example, which is of general illustration for this problem. Moeschberger and Klein (1995) presented a comprehensive survey on the dependent competing risks.

3.3. Some of the Recent Advances.

Pintilie (2007) raised a very interesting but perhaps somewhat counter-intuitive point in regards to competing risks analysis. The argument Pintilie (2007) made was that in some cases, depending on the objective of the experiment, it may be useful to ignore competing risks, as long as the results are interpreted properly. The author illustrated the issue with a fictional example about cancer occurrences in two towns. We found that her example could be easily transformed into a research problem in the field of reliability of computer networks. We present the network reliability example as follows, but the original ideas and arguments should be credited to Pintilie (2007) since there is nothing innovative in our transformation of the example.

Assume a computer science graduate student is conducting a study to compare the vulnerability of two networks of the same magnitude. One network consists of nodes running OS-W, and the other of nodes running OS-U. Both networks are very similar in operating environments, size, etc and the differences are negligible. The student formulates two hypotheses to test: (i) Is there significant difference in the incidence of node failures caused by security compromises between two networks? This is intended to determine if one of the networks would require more attention from the network administration team. (ii) Is there a difference in the rate of failure due to security compromises between the two networks? This is intended to test which of the OSes is more vulnerable to security breaches.

The competing risks in this case can be the hardware failures. Furthermore, the hardware for both OSes is different, and their reliability may be different too. In the testing of the first hypothesis, following Pintilie's (2007) logic, the failure from the competing risks should be considered, since if a node failed due to hardware failure before being exploited, the failure caused by hardware would reduce the exposure of the OS to hackers. Therefore, in the first testing case, the competing risks should be considered, and this is called *analysis of the hazard of sub-distribution*. However, in the second test case, the goal is to test whether one OS is more vulnerable than the other in terms of the failure rates due to security breaches. This is ideally done in the virtual scenario where the competing risks such as hardware failures do not exist. Therefore, it is desirable to exclude the competing risk events. This is called the analysis of the *cause specific hazard* (CSH) and is performed in the absence of competing risks events.

After the problem formulation, similar models to Pintilie's (2007) can be used to analyze the experiment data and build corresponding models. There were also follow-up comments and reply about Pintilie's (2007) paper, by Latouche et al. (2007).

Pintilie's (2006) recent monograph, *Competing Risks: A Practical Perspective*, contains an excellent overview of competing risks analysis. Furthermore, the monograph provides relatively detailed instructions to perform competing risks analysis with two of the most widely used software package, *R* and *SAS*.

4. IDENTIFIABILITY ISSUES.

Up to this point, we largely follow David and Moeschberger (1978) notation system. What David and Moeschberger (1978) called *theoretical failure times*, Y_i ($i = 1, 2, ..., k$) is also known as *latent failure times*. All but the smallest of Y_i is actually observable for a particular cause. Once the system has failed, the smallest Z=min $\{Y_i\}$and its associated risk is identified as the system failure time and cause, and the remaining theoretical or latent failure times lose the meaning (Crowder 2001).

There is a potential disturbing issue with the latent failure times approach, the identifiability issue (David and Moeschberger 1978, Crowder 2001). Simply put, the distribution of observed lifetimes X_i ($i = 1, 2, ..., k$) is completely determined by the equation (34), the joint distribution of the latent failure times. However, the inverse is not necessarily true. That is, the distributions of X_i do not uniquely identify the joint distributions of Y_i. What is more disturbing is that the distribution of X_i (e.g., equation 33) can always be represented in the form of equation (21) (which assumes independent) by proper picks of independent variates Y_i' having p.d.f. $p_i'(y)$. In other words, the dependent risk model with joint *p.d.f.* $p(y_1, y_2, ..., y_k)$ [*p.d.f* in (34)] is indistinguishable from the independent risk model $\prod_{i=1}^{k} p_i'(y_i)$. Obviously, if the risks are indeed independent, there is no issue; otherwise, the statistical results such as (21) may mislead the whole analysis.

Some researchers consider that the identifiability issue is resolvable as long as caution is taken. They believe if a specific functional form is assumed for the joint *p.d.f* of $Y_1, Y_2, ..., Y_k$, then, generally, the model can be fully identified by estimating the model parameters with the likelihood function, and the independence assumption can be tested (David and Moeschberger 1978). Some other researchers are more pessimistic, and this has been the major point where controversy on competing risks analysis arises.

This identifiability issue was first noted by Cox (1959) and further extended by Tsiatis (1975). Crowder (2001) called the results Cox-Tsiatis impasse and expressed it as the following proposition, of course with Crowder's notation introduced in section 2.2. Suppose that the set of survival functions S (j, t) is given for some model with dependent risks. Then there exists a unique proxy model with independent risks yielding identical $S(j, t)$. It is defined by $S^*(t) = \prod_{i=1}^{k} S_i^*(t_i)$ where $S_i^*(t) = \exp \int_0^t h(i,s)ds$ and the sub-hazard function $h(i,s)$ derives from the given $S(j, t)$.

The proof of the proposition (omitted here) reveals that the proxy model is constructed by using the original sub-hazards $h(i, t)$ for the $h_i^*(t)$. If the original model has the independent risks, then it is its own proxy, because in this case $h(i, t) = h_i(t)$, and so $S(t) = S^*(t)$. The troubling part is that this hazard condition can hold even without independent risks (Crowder 2001). In addition, while each dependent-risks model has a unique independent-risk proxy model, each independent-risk model has a whole class of satellite dependent-risks model and that this class can be further partitioned into sets with the same marginals (Crowder 2001).

Furthermore, the issue of identifiability persists in the less-distribution or semi-parametric competing risks analysis approaches. As we will discuss in the next section, the so-called mixed proportional hazard model (MPH), which is an extension of Cox proportional hazard model, assumes that the hazard rates of the latent failure times depend multiplicatively on the elapsed duration, observed regressors (co-variates) and unobserved heterogeneity (the frailty components). One example of the so-called frailty could be the economic conditions of a group of patients under observations, the differences between patients in economic conditions exist objectively, but not observable, perhaps due to the restriction of laws or whatever other reasons. If the unobserved determinants, such as frailties are dependent across the risks, then the failure times are dependent given the regressors (co-variates). Heckman and Honore (1989) showed that the MPH is identified if there is sufficient variation in the latent failure times with the regressors. Here, the identifiability is a little bit different from the previous one since the MPH is a nonparametric model and no distribution is assumed. Instead, the identifiability in the MPH is concerned with invertibility of the mapping from the model determinants to the distribution of (T, C), T is set of the observed failure times and C is set of corresponding causes. The identifiability issue is important because it implies that the estimates of model specification are not completely driven by parametric functional form assumptions on the model determinants. Abbring and van den Berg (2003) further relaxed the identifiability conditions set by Heckman and Honore's (1989).

The identifiability issue may be further complicated by an even broader issue in the parametric model selection for the observed failure times. As explored by Marshall et al. (2001), the issue is how does a researcher know Weibull distribution is more appropriate than, for example, Gamma distributions? In situations when there is not the identifiable "signature" mechanism, for example, the "lack of memory" property of exponential distribution, the choice of distribution naturally rests on the results from fitting the distribution models. There are generally three statistical methods for evaluating the fitting results (Marshall et al. 2001). The most popular method is the goodness-of-fit test. The second approach is to test the hypothesis that the chosen family is correct against the alternative hypothesis that a second specified family is correct and it treats the two families asymmetrically. The third approach is to choose two or more possible candidates and then use the data to select the most appropriate candidate. The difference between the second and third approaches is that all candidates are treated equally in the latter. The third approach then requires the procedure to evaluate the different candidates. There are two criteria that are often used: maximum likelihood and minimum Kolmogorov distance, and the hybrid of both are also used. Marshall et al (2001) used Monte Carlo simulation to generate data from known distributions and then used the third approach to investigate whether or not the data can "recognize" its parent distribution where they were generated from. They limit the study to the non-negative survival distributions including Weibull, gamma, lognormal, and geometric extreme exponential. What they found is that neither of the maximum likelihood and minimum Kolmogorov distance performs uniformly best. Since all but the lognormal distributions have the exponential distribution as special cases, what is interesting is that when the data is generated from exponential distribution, these models with exponential distribution as their special cases become less certain, for example, and may be strongly influenced by sample size. The issue becomes even more interesting with the so-called "rich" distribution. For example, the data from gamma distribution may be fitted by a Weibull distribution better than by its parent distribution. Weibull distribution is then said to be "richer" than gamma in some range of their parameters. Therefore, caution should be taken particularly when the tail behavior of the distribution model is important, since extrapolation to regions of little or no data may lead to wrong inferences (Marshall et al. 2001).

5. SUMMARY ON PARAMETRIC APPROACH.

Before starting the discussion of less-distribution dependent approaches, it might be helpful to have a summary of the distribution-dependent approach or parametric approach, especially after the discussion of the identifiability issue. The parametric approach is based on the concepts of theoretical or latent failure times. The latent failure times (Y_i) are multivariate, but the observed failure times (X_i) is univariate as formulated in David and Moeschberger's (1978) notations. The probability distributions of Y and X and the estimations of the distribution parameters form central tasks for competing risks analysis. This distribution parametric approach has been used historically; from this perspective, it can be said that competing risks analysis is independent of modern survival analysis. As we will see, the less distribution dependent or distribution-free, which is also called semi-parametric or non-parametric approaches respectively, draw extensively from modern survival analysis.

As indicated by Crowder (2001), another way to look at the latent failure time distribution approach is that it attempts to infer marginal distributions from observed data that is a consequence of competing failure risks acting together. In other words, it attempts to estimate the net risks from observations on crude risks. Due to the identifiability problems discussed above, the inference of marginal survival function from the latent failure times survival function cannot be done unless the risks are independent or some restrictions are made on the marginal survival functions. Without the restrictions, the best that can be obtained is some bounds of net failure probability (Crowder 2001). For this kind of limitation, plus the question of whether it even make sense to focus on the estimation of marginal survival function, the traditional competing risks analysis we introduced so far has received strong criticisms from some statisticians, especially from statisticians in survival analysis field. For example, it is argued that a doctor would normally just give advice on the illness in question, rather than worrying about other minor or incontrollable risks his or her patient may experience (such as being hit by a bus) (Crowder 2001). This suggests to focus on major failure risks and make inferences free of minor risks.

While those criticisms are certainly valid in biomedical fields, our opinion is that the situation may be different in reliability analysis. For example, when an airplane crashes, any information that helps to identify the cause of failure is extremely valuable. The case of an airplane investigation is probably typical for any component-based engineered systems. It is understandable that competing risks analysis, the applications of which are dominantly used in biomedical and actuarial sciences, has currently shifted away from the traditional latent failure time distribution approach, especially given the complexity of the approach and its limitations. However, we believe that the approach is very valuable in engineering reliability analysis and more research efforts are desirable to overcome or alleviate the existing problems associated with it. That also explains why we allocate significant space to discuss this approach, despite it appears out of favor in recent literature of competing risks analysis.

6. Semi-Parametric and Non-Parametric Approaches

In this section, Crowder's (2001) notation system and formulation are used, and many of the results are drawn from this reference.

6.1. Proportional Hazards

To introduce semi-parametric and non-parametric approaches, we need a few additional concepts. Recall the *overall hazard function* [$h(t)$] and *sub-hazards function* [$h(t, j)$] defined in equation (17) and (18). If the relative risk of failure from cause j at time t, $h(j,t)/h(t)$, is independent of t for each j, then the risks are said of proportional hazards. In other words, as time goes on, the relative risks of various causes do not change. No one gains or loses its share (proportion) of the overall risk, although the total risk may be up or down (Crowder 2001). The following theorem, recognized by several authors since the 1970s in various studies, confirms that this proportionality is another version of the independence of cause and time of failure.

Crowder (2001) expressed the theorem as the equivalence among the following three assumptions: (i) proportional hazards; (2) the time and cause of failure are independent; (3) $h(j, t) / h(k, t)$ is independent of t for all j and k. In this case, $h(j, t) = p_j f(t)$ or $F(j, t) = p_j F(t)$, where $p_j = \Pr(C = j) = F(j, \infty) = S(j, 0)$ is the marginal distribution of C in Crowder (2001) notation.

The proportional hazards described here are different from that in univariate survival analysis, although they are similar. Actually, we will review the extension of the proportional hazards in univariate survival analysis to the competing risks analysis in section 6.2.

In addition, one needs another type of hazard function, $g(j, t)$, which is conditional on both $C=j$ and $T>t$, rather than conditional on $T>t$ only.

$$g(j,t) = f(j.t)/S(j,t) = -d\log S(j,t)/dt \qquad (39)$$

This is the hazard for failures due to the cause j only, and it arises naturally in the development of competing risks analysis theory. In practice, when the proportional hazards do not hold, the so-called piece-wise proportionality may be used (Chiang 1960, 1970, 1991).

6.2. Regression Models

In most failure time analysis, the *i.i.d* (independent and identical distribution) assumption is not satisfied. A regression model that relates the covariates with the distribution parameters is built to capture the influences of covariates. A similar approach is used in non-parametric modeling, but, in semi-parametric modeling, the form of distribution is assumed rather than totally distribution-free.

6.2.1. Proportional Hazards Model

The univariate proportional hazards model by Cox (1972) can be extended to the competing risks modeling. A natural extension can be made in terms of the sub-hazard functions:

$$h(j,t;x) = \Psi_{jx} h_0(j,t) \quad (40)$$

where Ψ_{jx} is a positive function of the vector of covariates x for cause j. $h_0(j, t)$ is some baseline hazard function.

As mentioned previously, the "proportional hazards" has different meanings in the univariate survival analysis and the competing risks analysis. Now if one imposes the "proportional hazards", in the context of competing risks analysis, on the model (40), that is, replacing $h_0(j,t)$ by $p_j h_0(t)$:

$$h(j,t;x) = \Psi_{jx} p_j h_0(t) \quad (41)$$

Ψ_{jx} can take different forms, for example, $\Psi_{jx} = \exp(x^T \beta_j)$. Another way to look at the semi-parametric and non-parametric approaches is the specification of $h_0(.)$; $h_0(.)$ could be specified as some distribution such as Weibull, or left unspecified and computed by procedure such as maximum likelihood from observation data.

6.2.2. Accelerated Failure Time model.

Similar to the Cox proportional hazards model, the accelerated failure time model in univariate survival analysis can be extended to the competing risks analysis. When the effect of covariates vector x is to accelerate the time scale t by a factor Ψ_{jx} in the baseline model $S_0(j, t)$,

$$S(j,t;x) = S_0(j, \Psi_{jx} t) \quad (42)$$

A second proportional hazards in the context of competing risks can be incorporated into (42) and the model becomes:

$$S(j,t;x) = S_{00}(\Psi_{jx} t)^{p_j} \quad (43)$$

The p_j is the marginal distribution of C in the Crowder (2001) formulation. One example of Ψ_{jx} is $\Psi_{jx} = \exp(x^T \beta_j)$.

The idea of accelerated failure tests was originated from engineering reliability, and they are conducted to quickly obtain failure data of products under "accelerated failure conditions" such as exposure to excessive stress. The obtained data are used to parameterize accelerated failure time models, such as (42) and (43), which are then used to extrapolate the reliability under normal operation conditions. Escobar and Meeker (2006) presented a very comprehensive review on this topic.

6.2.3. Proportional odds model.

The univariate survival analysis model for proportional odds is:

$$[1 - S(t;x)]/S(t;x) = \Psi_x [1 - S_0(t)/S_0(t)] \quad (44)$$

A natural counterpart in competing risks is:

$$[1 - S(j,t;x)]/S(j,t;x) = \Psi_{jx} [1 - S_0(j,t)/S_0(j,t)] \quad (45)$$

The second stage extension of proportional hazards in the context of competing risks yields:

$$[1 - S(j,t;x)]/S(j,t;x) = \Psi_{jx} \{1 - [S_{00}(t)]^{p_j} / [S_{00}(t)]^{p_j}\} \quad (46)$$

that is, replacing $S_0(j,t)$ by $[S_{00}(t)]^{p_j}$.

6.2.4. Mean residual lifetime model.

The mean residual lifetime at age t is defined as:

$$m(t) = E(T - t) \mid T > t) \quad (47)$$

It is the expected time left to an individual that has survived to time t. The corresponding life expectancy is $E(T|T>t) = m(t)+t$.

It can be derived (Crowder 2001):

$$m(t) = \int_t^\infty (y - t) f(y \mid y > t) dy$$
$$= \int_t^\infty (y - t)[f(y)/S(t)] dy = \frac{\int_t^\infty S(y) dy}{S(t)} \quad (48)$$

The inverse relationship can be derived as:

$$S(t) = \frac{m(0)}{m(t)} \exp\left(- \int_0^t m(y)^{-1} dy \right) \quad (49)$$

For regression in univariate survival analysis, the mean residual lifetime can be regressed with a covariate vector x.

$$m(t;x) = \Psi_x m_0(t) \quad (50)$$

m_0 is some baseline mean residual lifetime function and Ψ_x is a positive function of covariates vector, for example, $\Psi(x) = \exp(x^T \beta)$.

When extended to competing risks, (50) can take the form:

$$m(j,t;x) = \Psi_{jx} m_0(j,t) \quad (51)$$

The cause-specific mean residual lifetime is defined as (Crowder 2001):

$$m(j,t) = \int_t^\infty (y - t) f(j, y \mid y > t) dy = S(t)^{-1} \int_t^\infty S(j, y) dy \quad (52)$$

The above brief introduction might present an impression that the extensions from univariate survival analysis are straightforward. There is enormous complexity hidden: one has to justify the assumptions in the context of real applications, and there is also the necessity to accommodate observation censoring.

7. MULTIVARIATE COMPETING RISKS ANALYSIS

The term multivariate competing risks analysis was proposed recently by Wohlfahrt et al (1999) in an application-oriented study. One would naturally think the multivariate competing risks analysis should be the multivariate extension of standard competing risks analysis. It appears that the extension has not received full recognition yet. Before introducing Wohlfahrt et al (1999), let us see how competing risks analysis is approached by the researchers in multivariate survival analysis.

In the competing risks analysis we discussed so far, there is only one real or observable failure event. Although there is the notion of latent failure times variables (which are not observable), it makes some sense to treat competing risks analysis as univariate, given that only one of them can actually be observed. As we have seen, the two major complications from competing risks analysis are: (i) the dependence between the competing risks; (ii) identifiability issues.

Hougaard (2000) criticized the competing risks analysis, especially the latent failure time approach; understandably, the identifiability issue was his main concern. Hougaard (2000) also indicated that the standard frailty approach in multivariate survival analysis does not provide a solution to competing risks analysis either. He does consider that cause-specific hazard functions can be estimated, for example, with Nelson-Aalen estimator (e.g., Nelson 1969), or include covariates in a Cox model. The other problem Hougaard (2000) addressed regarding competing risks analysis is that in some cases, it is difficult to classify the events, or identify the cause of failure. This latter is treated as censoring of failure mode in competing risks analysis, similar to the way censoring time was treated in survival analysis.

The different opinions in survival analysis and competing risks analysis might be more related to the dominant problems each field needs to address. The dominant applications of survival analysis are in biomedicine, where one may focus on a single dominant failure cause. In medical clinical trials, an experiment may only need to evaluate the effects of a single drug or treatment, and the other factors can be treated as covariates. Therefore, in many occasions, there is a single dominant failure event in biomedicine. From this perspective, survival analysis is certainly not a "*lost cause*". On the other hand, the dominant applications of competing risks analysis are in the actuary and public health, in which multiple risks are the norm rather than the exception. An insurance company has to determine various premiums for different insurance options. Therefore the effects of removing a risk from consideration is of extreme importance, since it has to estimate how much it needs to charge extra to cover a specific source of risks to be profitable. In the public health field, Bernoulli's question of how the elimination of smallpox risk would affect the overall population mortality is still relevant today. It is true that we still lack effective solutions to answer Bernoulli's question, but in many cases, competing risks analysis offers feasible and relatively simple solution. To reconcile both perspectives, Bedford's (2005) suggestion that the validation of competing risks analysis models has to be *softer* (i.e., resting on the evaluation of the engineering and organizational context), might be helpful.

It is our opinion that reliability analysis, with broader target systems, may face problems similar to both biomedicine and actuarial science and should try to draw the best from all three fields: survival analysis, multivariate survival analysis, and competing risks analysis. The simplest reason is that survival analysis provides basic models for reliability analysis. The competing risks analysis and multivariate survival analysis are needed to address series and parallel systems, respectively. In addition, any advances in handling dependent risks are always needed in reliability analysis.

We now turn to Wohlfahrt et al's (1999) multivariate competing risks analysis concept and model. They presented a motivating example with a breast cancer follow up study. The objective of their study was to investigate whether a woman's number of births, besides being an important risk factor for breast cancer as such, was also predictive of disease severity at diagnosis, in order to select women for a targeted cancer screening. The analysis was conducted to compare the effects of birth number on the incidence of breast cancer according to two measures of severity at diagnosis: tumor size and number of positive nodes. Therefore, there are two types of events in observations: one is the tumor size, and the other is the number of positive nodes. One may argue that those are just two symptoms of the same cancer event, but the authors argued that the two classifications do exist separately at diagnosis.

The first key element is that there are *two or more types of failure events* associated with each individual in observation. Now, there are covariates such as the number of births and other factors that may affect each type of the observed events. For example, the number of births has strong effects on the tumor size, as well as on the positive nodes. What Wohlfahrt et al. (1999) were particularly interested in was the hypothesis that both types of events might be strongly dependent, as small tumors tend to be node-negative. In other words, one may speculate that the two findings may reflect the same phenomenon.

Obviously, one may easily construct a similar application in reliability analysis. For example, a network administrator may be interested in observing two types of security events: (1) *virus* infection and spread; (2) the system *instability,* which may be caused by, for example, the operating system memory leak which may be time dependent. Both types of events are not fatal and may be observed simultaneously. Still, other types of events may be observed (for example, hardware failure and/or malicious attacks). In the other dimension, there are multiple risks such as efficacy of the virus protection program, the operating systems patch level, hardware wearing, and hacking activity. Of course, the observed events are very likely dependent on each other. For example, a virus-infected node would be more vulnerable to hacker's attacks, and it might even be the case that a hacker uses a virus as the initiating attacking tool. Similarly, one would like to test the dependence between events and whether or not the effects of covariates on instability are simply a consequence of its influences on virus infection. In other words, one may wish to know, whether the instability is more the consequence of virus infection or the consequence of its own problem—the memory leak of its running operating systems.

We summarize the modeling framework proposed by Wohlfahrt et al (1999) for their *multivariate competing risks analysis* based on the Cox proportional hazard model.

The standard univariate Cox regression model is:

$$\lambda(t) = \lambda_0(t)\exp(\beta x) \tag{53}$$

where t represents time or age and x being the covariates vector. β is a vector of the parameters corresponding to each covariate in the covariates vector.

The version for multiple events would be:

$$\lambda_{jk}(t) = \lambda_{0jk}(t)\exp(\beta_{jk} x) \tag{54}$$

j=1, 2, ..., J belong to one classification of J subtypes, k=1, 2, ..., K belong to the other classification of K subtypes. The effects of covariates are different for each classification of subtypes, and are reflected by β_{jk}. The model (54) is the straightforward extension of (53), and it considers the cross product of both classifications.

Wohlfahrt et al (1999) considered a more parsimonious log-additive model:

$$\lambda_{jk}(t) = \lambda_{0jk}(t)\exp(\beta^0 x + \beta_j^1 x + \beta_k^2 x) \tag{55}$$

where the effect of covariates x is log-additive on both subtype classifications.

Wohlfahrt et al (1999) claim that equation (55) offers a natural means for testing the lack of difference in effects, according to one subtype classification. That is, testing the models: $\lambda_{jk}(t) = \lambda_{0jk}(t)\exp(\beta^0 x + \beta_j^1 x)$ or,

$\lambda_{jk}(t) = \lambda_{0jk}(t)\exp(\beta^0 x + \beta_k^2 x)$.

These models for $\lambda_{jk}(t)$ are called *multivariate competing risks* models since they can be applied for analyzing two or more sets of competing risks, making it possible to test hypotheses about the multivariate effects of risk factors.

8. Competing Risks Analysis as a Multi-state Model.

While the framework of multivariate competing risks analysis such as Wohlfahrt et al (1999) seems to receive little attention, the problem of an individual experiencing multiple events, each of which might have multiple competing risks or all of which have a common set of competing risks, does exist widely in many applications. What we discuss in this section is a recent attempt to address the same problem discussed in the previous section with the so-called *multi-state survival analysis* model.

Multi-state model is a rapidly expanding topic and a recent issue of *Statistical Methods in Medical Research* is dedicated to the multi-state model (Andersen et al. 2002, Andersen 2002, Putter 2007). In the following, we briefly discuss the formulation of the problem of competing risks analysis with multi-state model.

Competing risks analysis deals with one initial state and several mutually exclusive absorbing states. In the multi-state model, the states are not exclusive and there are intermediate or transient states. To simplify the model, one may designate a single start state and a single unique absorbing state. Essentially, the multi-state model is an integration of Markov chains with survival analysis, and the challenges lie in forming a state-transition model that conforms to the property of Markov chains or semi-Markov chains. The remaining work is simply standard Markov chain analysis. Of course, the Markov chain has been an important tool for reliability for quite a long time. This is a field where reliability analysis may actually be ahead of survival analysis and competing risks analysis. For the status of multi-state model in reliability analysis, one may refer to Lisnianski and Levitin (2003) and several others.

We draw the following introduction from Putter et al. (2007) on how the competing risks analysis is formulated as a multi-state model, or we simple call it a Markov chain model.

Let T denote the time of reaching state j from state i ($i \to j$), the hazard function (transition probability in terms of Markov chains) is defined as:

$$\lambda_{ij}(t) = \lim_{\Delta t \downarrow 0}\left(\frac{\Pr(t \le T < t + \Delta t \mid T \ge t)}{\Delta t}\right) \tag{56}$$

The cumulative hazard for transition $i \mapsto j$ is:

$$H_{ij}(t) = \int_0^t \lambda_{ij}(s)ds \tag{57}$$

There are two essential issues that have to be addressed: the defining of time scale and the compliance with the Markov chain properties. There are two approaches to define the time scales. One is the so-called *clock forward* scale that refers to the time since the individual entered the initial state. The clock keeps ticking forward for the individual continually, even when it travels through the transient states. The other scale is called "clock reset" referring to the time since the entry into the state; the clock is reset to 0 each time the individual enters a new state. The clock reset scale is also known as *backward recurrence time*. The decision to adopt one over the other lies in the consideration of their potential influences on the properties of Markov chains.

Several kinds of violations or relaxations are possible and dependent on the specific problem in study. One potential violation of Markov property is a situation where the order of states visited influences transition probabilities. One solution for this violation is to split a state into two or more states, i.e., treating the different visiting sequence as different transitions. A second relaxation of the Markov assumption is to let the sojourn times (as covariates) depend on the times at which earlier states have been entered. The resulted model is called *state arrival extended semi-Markov chain* model, which means that the $i \rightarrow j$ transition hazard or probability depends on the time of arrival at state i.

The transition hazards functions (56) can be extended with Cox regression approach, that is, including covariates effects and the covariates can be time-dependent. For example,

$$\lambda_{ij}(t \mid x) = \lambda_{ij,0}(t)\exp(\beta_{ij} x) \quad (58)$$

where $\lambda_{ij,0}(t)$ is the baseline hazard for transition $i \rightarrow j$ and β_{ij} is the vector of parameters that describe the influences of covariates (x) on transitions. The follow-up to (58) should be obvious to researchers in reliability theory, since the Markov chain has been used in reliability analysis extensively. Nevertheless, it seems that the integration of competing risks analysis and survival analysis indeed brings about some very useful new insights.

It seems that the extension of either standard survival analysis or competing risks analysis to multivariate or multi-state modeling framework is inevitable. In an extension of survival analysis for modeling insect population dynamics, Ma (1997) adopted the Leslie matrix model that uses the survival functions for insect development and survival as matrix elements. The Leslie matrix is essentially a Markov chain model with the survival and developmental probabilities as the elements of the matrix. Therefore, the integration of survival analysis and Markov chain seems to be a very natural modeling strategy (Ma 1997)

9. Current Application Status in Computer Science and IEEE Related Engineering.

9.1. Applications Found in IEEE Digital Library.

We conducted an online search in IEEE Digital library with the keyword "competing risks" in October of 2007, and found only about 20 relevant papers. In this section, we briefly review each paper found from the search to get a glimpse of the status of competing risks analysis in computer science and IEEE related engineering fields.

Ishioka and Nonaka (1991) presented a maximum likelihood estimator (MLE) for a system consisting of two series components, whose lifetimes follow Weibull Distribution. The independence seems implicitly assumed. It is not clear how the identfiability issue was dealt with. Dognaksoy (1991) applied competing risks analysis to analyze *masked* system failure data. Masked failure is the failure whose cause is not identified. The exact estimation of the confidence interval (CI) of failures from observed interval is not possible. The masked failure was treated as censored observations in the study, and the independence between the multiple modes was implicitly assumed. Menon et al (1991) applied the competing risks analysis to the quality assurance. The VLSI failures of interconnects were modeled with competing risks analysis. Whenever defects exist, the failure distribution is bimodal. When there are no defects, the failure distribution follows the lognormal. This multimodality in failure distribution, as an indication of competing risks, may be useful for general quality testing; however, this only works when the failure modes are independent.

Monitoring and inspection based maintenance schemes also need to consider the effects of competing risks. Coolen and Dekker (1995) concluded that once the cost-effectiveness of monitoring-based maintenance was established, the competing risks were irrelevant in determining the optimal monitoring interval. This might be an exception rather than the rule. Nevertheless, the study demonstrated that competing risks analysis can be applied to the optimization of maintenance policy-making. We think that competing risks may actually have significant influences on maintenance policy. Usher (1996) presented a scheme to predict the components' reliability from the system reliability data. The motivation was that the reliability predicted in this way is more realistic than the reliability data from isolated component tests. That is, we build a system of series components and observe system failure and the cause of failure, then try to infer the component reliability or marginal failure probability. The mask or censoring is a major challenge in this approach. Of course, if the failure modes are dependent, the identifiability issue emerges. Papadopoulos et al. (1996) applied hierarchical Bayesian approach to estimate the parameters of multinomial, multivariate exponential, and Marshall-Olkin multivariate exponential distribution.

Sun and Tiwari (1997) proposed a nonparametric hypothesis test statistic for the possible dependence between the different failure risks. Reineke et al. (1999) assigned a competing risks model for each of the five subsystems of a bridge reliability structure. Each subsystem or competing risks model had two failure modes. It was assumed that the expected system lifetime and maintenance actions were known. Their study tried to identify the optimal preventive maintenance. Cooke and Bedford (2002) advocated the adoption of competing risks analysis as a general mathematical model for failure modes analysis in the context of building the reliability databases. They argued that reliability databases should not only store data that can be used to derive failure probability, but also the causes of failures. This makes competing risks analysis necessary. Bunea and Bedford (2002) adopted competing risks analysis to assess the operation reliability of nuclear facility maintenance. Nuclear facility maintenance data is highly right-censored due to the preventative maintenance. They introduced three competing risks models: independent failure risks models, strongly correlated censoring and dependent failure risks models. The challenge is that it is not possible to identify the accurate failure modes due to the censoring. They call the inability to identify accurate failure modes *the uncertainty of models*. What they found is that the model uncertainty does have significant implications to optimal maintenance plans. The solution they suggest is to use expert judgment to quantify the dependence between the competing risks, and then the model uncertainty is incorporated into the optimization of maintenances.

Park and Kulasekera (2004) developed maximum likelihood estimators for the competing risks analysis of data from multiple groups, with both failure time and failure cause censorings under multivariate exponential distributions. Park (2005) also studied the parameter estimation in competing risks analysis with the EM (expectation maximization) algorithm. Kundu and Sarhan (2006) extended Park and Kulasekera (2004) by assuming Weibull distribution failure times, rather than the exponential distribution. They also tried EM algorithms for the parameter estimations. Sarhan (2007) also studied the maximum likelihood estimation for the competing risks models with lifetimes following the generalized exponential distribution.

Zhao and Elsayed (2004) formulated a competing risks analysis model consisting of two types of failures. One failure risk is the soft failures due to degradation, and the other is the hard failures due to catastrophic events. The degradation was modeled with a Brownian motion process and the first arrival to a boundary was treated as a soft failure. The hard failures, due to catastrophic events, were described with a Weibull distribution. Pascual (2007) studied the accelerated failure test planning under multiple competing risks and the distribution of the failure times was assumed to follow the Weibull distribution. Dimitrakopoulou et al. (2007) introduced a three-parameter failure time distribution with bathtub and upside-down bathtub shaped failure rates. Weibull distribution can be derived as a special case of the three-parameter distribution. The authors also offered competing risks failure interpretation for the derivation of the distribution.

9.2. Selected Papers Found in MMR-2004.

In the following, we briefly review a few survival analysis related studies presented in a recent International Conference on Mathematical Methods in Reliability, MMR 2004 (Wilson et al. 2005).

Bedford (2005) presented a comprehensive review on the key issues of competing risk modeling in the context of reliability. His discussion is particularly relevant to maintenance problems. Bedford treated the competing risk problem as arising from a renewal process, in which only the first possible event, occurring after renewal, is observable. He emphasized the problem, which he referred to as the inability to infer marginal distribution without invoking un-testable distributional assumptions. What Bedford referred as *competing risks problem* is actually the identifiability issue we discussed previously, which has been the major controversial issue in competing risks analysis. Bedford's review is particularly inspiring by treating the identifiability problem as a more general issue of model identifiability. Selecting more tight class is helpful for identifying more specific model in the class; however, the extreme parsimony in this strategy may lead to failure to capture the features in the data. Bedford's (2005) recommendation is that the balance should be achieved by better understanding the engineering context while specifying tight families of models. The other issues Bedford (2005) addressed include independent and dependent competing risks, characterization of possible marginals, Kolmogorov Smirnov test, the bias of independence, maintenance as a censoring mechanism, and relaxation of the renewal assumption.

Bunea and Mazzuchi (2005) offered an example of the accelerated failure modeling with dependent competing risks (failure modes) and the study reveals the high model sensitivity to the degree of dependence between the competing risks.

Dewan and Deshpande (2005) reviewed distribution-free test statistics for bivariate symmetry, F(x, y) = F(y, x), for all (x, y), where x and y are latent failure time random variables associated with two competing risks and F(x, y) is their joint distribution. The rejection of the hypothesis indicates that one risk dominates the other. They also reviewed the statistics for testing the independence between failure time (T) and failure cause (C). Both tests are of significant importance. For example, the independence between T and C implies that equation (12) $S(j,t) = \Pr(C = j, T > t)$ in Crowder's (2001) formulation can be simplified as $S(j,t) = \Pr(C = j)S(t)$. This will allow the study of failure causes and failure times separately.

10. PERSPECTIVE

In previous sections, we surveyed the major research in competing risks analysis and its applications in computer science and IEEE related engineering fields. It seems to us that the interaction of competing risks analysis with survival analysis, especially with multivariate survival analysis, is the trend, despite the occasionally contentious opinions between two fields. In this paper, we did not discuss survival analysis outside the context of competing risks analysis.

In section 9, we reviewed all 20 papers found (until the summer of 2007) from IEEE digital library with the keyword "competing risks". Although this search is certainly far from complete in the scope of scientific literature, it may demonstrate the status of the applications of competing risks analysis in computer science and IEEE related engineering fields. While the number of the papers is incomparable with the applications of competing risks in biomedicine, actuarial science, and public health, which are in the magnitude of hundreds or more per year, we have seen some excellent applications, especially in the last few years. However, it is also obvious that the potential of competing risks analysis should be much broad. Below, we suggest a few topics that have not been explored but with great promise in our opinion. Although we have not seen the applications of competing risks analysis in the suggested fields, they are certainly motivated by applications of competing risks analysis in other fields, especially by the applications published in IEEE digital library.

10.1. Reliability and Survivability of Computer Networks.

The reliability of computers and computer networks have been studied extensively and reviewed in several excellent monographs (e.g., Trivedi 1982, Shooman 2002, van Mieghem 2006). Survivability is closely related or even dependent on reliability. The key difference between survivability and reliability is that survivability is measured with the capability to endure catastrophic failure, which is often caused by malicious and unpredictable events (Krings 2007). Modeling of survivability has been very challenging for several reasons. First, unlike reliability, for survivability, we still lack a precise mathematical definition that is well accepted. Even if one devises a probability definition similar to reliability, the task to assign probability to the malicious intrusions is extremely difficult and unreliable. Secondly, survivability is more closely associated with economic values. Nobody cares about the survivability of a system that has little economic value. Reliability has a similar property, but survivability is much more direct. The extreme example would be if we were willing to invest enough resources, unpredictable attacks can be minimized. On the other hand, once the system is built, monetary investment often has little effect on the failures due to normal operation.

In addition, survivability also depends on maintainability, just as it depends on reliability. Maintainability is often modeled as an optimization problem with economic and environmental factors as constraints. Thirdly, it seems to us that survivability is also a competing risks problem, in the sense that these unpredictable events are competing risks for the system. In other words, a system under normal operation environments may fail due to either malicious events or normal failure mechanisms (such as wear-out or shock). An approach based on Survivable Network Analysis (Mead et al. 2000) is an example of a qualitative approach.

The second property suggests that game theory modeling should be useful for survivability modeling. Kumar and Marbukh (2003) have provided such an example. We suggest that evolutionary game theory can be more suitable for modeling survivability for the following reasons: (1) Rationality which is the basis of traditional game theory is far from realistic in malicious events. Evolutionary game theory is not based on the rationality assumption; instead, it is based on the dynamic interaction between the players, or it evolves the evolutionary stability strategy (ESS). ESS is the strategy that can resist both internal mutation and external invasion. (2) Survivability is highly dynamic, which again is consistent with the assumption of evolutionary game theory. (3) Evolutionary game theory can accommodate potentially infinite number of players, and it can be described with differential equations. This is particularly suitable for modeling computer networks, especially wireless sensor networks, with hundreds or even thousands of nodes.

The third property calls for competing risks analysis modeling. However, to the best of our knowledge, this has not been studied yet. There have been applications of various risks assessment modeling in survivability studies. However, none of the risks analysis methodology falls in the same category as discussed in this paper. We believe the competing risks analysis approach has potential to model reliability and survivability under a unified framework. This should be similar to Zhao and Elsayed (2004) did in accelerated failure test (AFT) planning, categorizing the failure types into two categories, one is the catastrophic failure and the other is degradation failure. As we suggested in the review of survival analysis (Ma and Krings 2008a, b), censoring can be used as a mechanism to model survivability. This is consistent since in univariate survival analysis censoring can be used as a catchall failure mechanism. Competing risks analysis should allow more flexible modeling of survivability, since there can be at least two failure mechanisms, plus the censor event.

However, the challenge of assigning probability does not disappear with the adoption of competing risks analysis. In this aspect, treating malicious events as random censored, such as we suggested in univariate survival analysis (Ma and Krings 2008a), could be advantageous. In addition, the integration with two other approaches may relieve the problem. One approach is the evolutionary game theory, and the other is the assigning of subjective probability based on expert opinions. Therefore, we expect that the integrated approach of evolutionary game theory, competing risks analysis, should be appropriate for modeling survivability. An additional advantage of this integrated approach is that economic and environmental constraints can be easily integrated into the modeling.

10.2. Software Reliability and Test Measurements.

As we suggested in the univariate survival analysis review article (Ma and Krings 2008a), survival analysis can be used to model software reliability. However, it may be necessary to develop new metrics or adopt existing ones such as Kolmogorov complexity (Li and Vitanyi 1997) to replace time variable in the survival analysis. Similar adaptation may also be needed in the application of competing risks analysis.

Sentas and Angelis (2005) applied survival analysis directly to the modeling of the software project duration, without transforming time random variables. This reminds us that other fields of software test measurements and project management also involve time-to-event random variables. Consequently, survival analysis and competing risks analysis should be ideal for analyzing the data from those time-to-event process. A natural question is: what are the advantages for adopting competing risks analysis or survival analysis over the classical statistics? One obvious example is that censoring is almost ubiquitous. The software is never bug free, and one very likely reason is that the debugging is *right-censored*. Without censoring, in other words, continuing the debugging until the discovery of the last bug, software should be as reliable as the mathematical algorithms. There are other advantages such as dealing with non-normal distributions, but the capability to deal with censoring is unique and fundamental.

Another follow-up question would be what are the advantages of competing risks analysis over univariate survival analysis? One may get the standard answer from the discussions in previous sections, such as multi-mode failure vs. single mode failure, handling dependency among failure risks, etc. When applied to software reliability, we suggest the following competing risks analysis model. Today's software, perhaps except for a few which is closely integrated with operating systems, is developed and tested relatively independent of operating system and network environment. The release of application software and operating systems may not be synchronized. Furthermore, the exploitations of vulnerabilities and patches make the synchronizations nearly impossible. Therefore, the failure of application software may be due to the failure of the package, or the operating system, or the malicious exploitation of either the packages or network security breaches. What makes the thing even more complicated is the dependence between these risks. Given the characteristics we describe here, a competing risk analysis model will be more advantageous than the univariate survival analysis.

As we argue in a separate article on multivariate survival analysis, the multivariate has advantages over univariate and competing risk analysis (Ma and Krings 2008b). One fair question would be, why do not we just use multivariate survival analysis as the single most useful approach? Our answer would be: there are niches for univariate, and competing risks analysis suits them perfectly. Generally, the complexity and power increases from univariate, competing risks analysis to multivariate survival analysis. As expected, there are costs associated with the power of the multivariate survival analysis. One is the complexity in both mathematical derivations and the statistical data modeling. Procedures for most univariate survival analysis and simple ones for the competing risks analysis are available in major standard software packages. However, this is not the case for multivariate survival analysis, which often has to be programmed, preferably with statistical software languages, such as R and S-Plus. The other challenge with multivariate survival analysis is that it requires more detailed data collection or experiment observations. Obviously, even if it is known that multi-modes and/or multiple failures are involved, but not observable for some reasons, and then it is not possible to apply multivariate survival analysis. A third potential challenge is that the model identifiability is much more complex in multivariate survival analysis, such as shared frailty models. This requires both high quality data and keen insights and experience from modelers, since in high-dimensions the intuitions and graphic representations are less helpful.

10.3. Prognostic and Health Management (PHM)

Similar to the arguments made in Ma and Krings (2008a) in the context of univariate survival analysis and Ma and Krings (2008b) in the context of multivariate survival analysis, we believe competing risks analysis should play important roles in PHM modeling of reliability, life predictions, failure analysis, quality control, risk assessment and predictions, etc. The most fundamental and unique advantage of competing risks analysis and survival analysis over the currently used approaches in PHM are their unique capability in handling information censoring. In PHM and other logistics management modeling, information censoring is a near universal phenomenon. Furthermore, both competing risks analysis and survival analysis are developed to analyze time-to-event random variables, of which failure events are the most straightforward and common. The "built-in" mechanisms in modeling failure events are obviously advantageous over the other emerging techniques such as artificial neuron networks (ANN), evolutionary computing, and Fuzzy logic. Competing risks analysis and multivariate survival analysis should also be highly valuable in analyzing various dependence events in PHM. Obviously, dependence exists as widely as the censoring in PHM modeling.

Keller-McNulty et al. (2006) equated *reliability* and *integrated system assessment*, which perhaps accurately reveals the scope of modern reliability analysis. Survival analysis and competing risks analysis provide powerful tools to model reliability as time-to-event random processes. Obviously, *reliability* or the *integrated system assessment* constitutes the core processes in prognostic and health management (PHM), competing risks analysis and survival analysis should become the standard toolkits for the PHM.

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