Deriving Equations

- $P_i(t+\Delta t) =$ probability of being in state S_i after Δt

$$P_i(t+\Delta t) = P_i(t)[1 - \sum_{i \neq j} \lambda_{ij} \ \Delta t] + \sum_{i \neq j} P_j(t) \lambda_{ji} \ \Delta t$$

as $\Delta t \rightarrow 0$

(differentiate)

$$\lim_{\Delta t \rightarrow 0} \frac{P_i(t+\Delta t) - P_i(t)}{\Delta t} = -P_i(t) \sum_{i \neq j} \lambda_{ij} + \sum_{i \neq j} P_j(t) \lambda_{ji}$$

Deriving Equations

- With m states => m differential equations
- m-1 independent equations

$$\frac{dP_1(t)}{dt} = \sum_{j \neq 1} P_j(t) \lambda_{j1} \quad - P_1(t) \sum_{j \neq 1} \lambda_{1j}$$

$$\frac{dP_i(t)}{dt} = \sum_{j \neq i} P_j(t) \lambda_{ji} \quad - P_i(t) \sum_{j \neq i} \lambda_{ij}$$

$$\frac{dP_{m-1}(t)}{dt} = \sum_{j \neq m-1} P_j(t) \lambda_{j(m-1)} - P_{m-1}(t) \sum_{j \neq m-1} \lambda_{(m-1)j}$$

- m^{th} equation $\quad 1 = \sum_{\forall k} P_k(t)$

Deriving Equations

- Matrix Notation

$$\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \frac{dP_i(t)}{dt} \\ \frac{dP_{m-1}(t)}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sum_{j\neq 1}\lambda_{1j} & \lambda_{21} & \lambda_{31} & \cdots & & \lambda_{m1} \\ \lambda_{1i} & \lambda_{2i} & \cdots & -\sum_{j\neq i}\lambda_{ij} & & \lambda_{mi} \\ \lambda_{1(m-1)} & \cdots & & & -\sum_{j\neq m-1}\lambda_{(m-1)j} & \lambda_{m(m-1)} \\ 1 & 1 & & \cdots & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_i \\ P_{m-1} \\ P_m \end{bmatrix}$$

Steady State Solutions

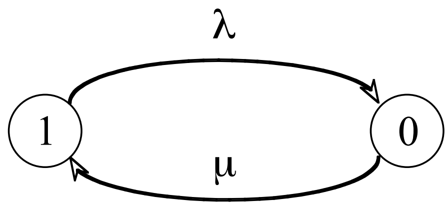
- Steady state solution:

$$\lim_{t\to\infty} \frac{dP_j(t)}{dt} = 0$$

- Steady state solution = Availability
  - set of linear alg. equations rather than linear differential equations

Steady State Solution

- Example: Simplex system with repair



λ = failure rate

μ = repair rate

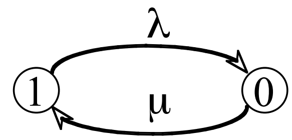
$$\begin{bmatrix} \frac{dP_0}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} -\mu & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

$$b = Ax$$

$$\begin{bmatrix} \frac{dP_0}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mu & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

Steady State Solution

- Simplex with Repair



- Solution:

$$P_0 = \frac{\lambda}{\mu + \lambda} \qquad P_1 = \frac{\mu}{\mu + \lambda}$$

- Steady State Availability

$$P_1 = \frac{\mu}{\mu + \lambda} = \lim_{t \to \infty} A(t)$$

- e.g.

$$\lambda = 10^{-3} \quad \Rightarrow \quad MTTF = 1000h$$
$$\mu = 10^{-1} \quad \Rightarrow \quad MTTR = 10h$$

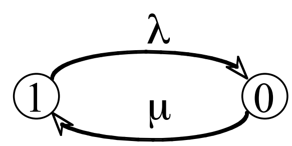
Availability:

The prob. that system is up

$$A = \frac{10^{-1}}{10^{-1} + 10^{-3}}$$
$$= 0.99 = 99\%$$

Transient Solution

- Simplex with Repair



$$\frac{dP_1(t)}{dt} = \mu P_0(t) - \lambda P_1(t)$$

with $P_0(t) + P_1(t) = 1$ we get

$$\frac{dP_1(t)}{dt} = \mu(1 - P_1(t)) - \lambda P_1(t)$$
$$= -P_1(t)(\mu + \lambda) + \mu$$

- $P_1'(t) + (\mu + \lambda) P_1(t) = \mu$ is a first order diff. equation

Transient Solution

- $P_1'(t) + (\mu + \lambda) P_1(t) = \mu$ has general solution

$$P_1(t) = \frac{\mu}{\mu + \lambda} + Ce^{-(\mu+\lambda)t}$$

- Get C by setting t=0

$$C = P_1(0) - \frac{\mu}{\mu + \lambda}$$

- Solution

$$P_1(t) = \frac{\mu}{\mu + \lambda} + \left(P_1(0) - \frac{\mu}{\mu + \lambda} \right) e^{-(\mu+\lambda)t}$$

Transient Solution

- with $t \rightarrow \infty$ we get

$$P_1(t) = \frac{\mu}{\mu + \lambda} + \left(P_1(0) - \frac{\mu}{\mu + \lambda} \right) e^{-(\mu+\lambda)t}$$

$$= \frac{\mu}{\mu + \lambda}$$

← our steady state solution (steady state availability)