Markov Process

- A stochastic process is a function whose values are random variables
- The classification of a random process depends on different quantities
  - state space
  - index (time) parameter
  - statistical dependencies among the random variables *X(t)* for different values of the index parameter *t*.

Markov Process

- State Space
  - the set of possible values (states) that $X(t)$ might take on.
  - if there are finite states => *discrete-state process* or *chain*
  - if there is a continuous interval => *continuous process*
- Index (Time) Parameter
  - if the times at which changes may take place are finite or countable, then we say we have a *discrete-(time) parameter* process.
  - if the changes may occur anywhere within a finite or infinite interval on the time axis, then we say we have a *continuous-parameter* process.

Markov Process

- In 1907 A.A. Markov published a paper in which he defined and investigated the properties of what are now known as Markov processes.
- A Markov process with a discrete state space is referred to as a *Markov Chain*
- A set of random variables forms a Markov chain if the probability that the next state is $S_{(n+1)}$ depends only on the current state $S_{(n)}$, and not on any previous states

Markov Process

- States must be
  - mutually exclusive
  - collectively exhaustive
- Let P_i(t) = Probability of being in state S_i at time t.

$$\sum_{\forall i} P_i(t) = 1$$

- Markov Properties
  - future state prob. depends only on current state
    - independent of time in state
    - path to state

Markov Process

- Assume exponential failure law with failure rate λ.
- Probability that system failed at $t + \Delta t$, given that is was working at time t is given by

$$1 - e^{-\lambda \Delta t}$$

with

$$e^{-\lambda \Delta t} = 1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \cdots$$

we get

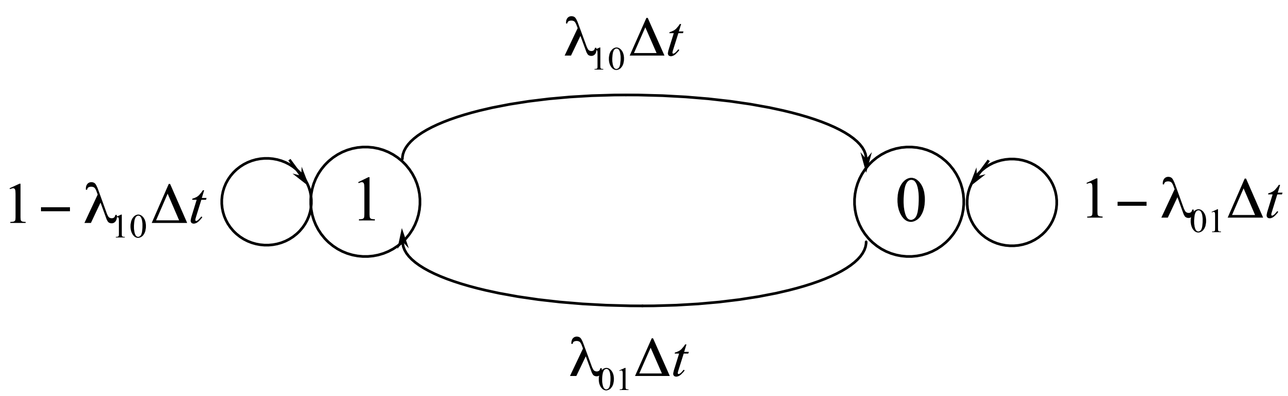
$$1 - e^{-\lambda \Delta t} = 1 - [1 + (-\lambda \Delta t) + \frac{(-\lambda \Delta t)^2}{2!} + \cdots]$$

$$= \lambda \Delta t - \frac{(-\lambda \Delta t)^2}{2!} - \cdots$$

Markov Process

- For small Δt

$$1 - e^{-\lambda \Delta t} \approx \lambda \Delta t$$



Markov Process

- Let P(transition out of state i in Δt) =

$$\sum_{j \neq i} \lambda_{ij} \Delta t$$

- Mean time to transition (exponential holding times)

$$\frac{1}{\sum_{j \neq i} \lambda_{ij}}$$

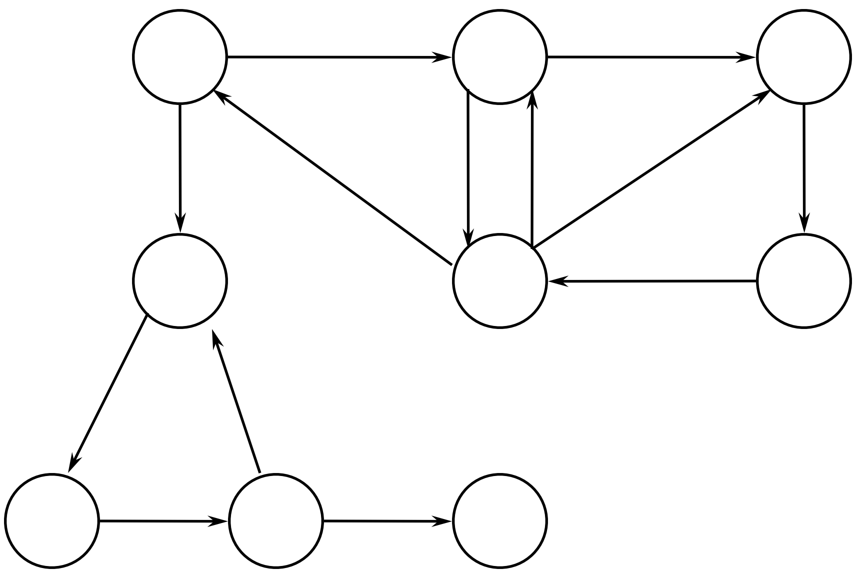
- If λ's are not functions of time, i.e. if $\lambda_i \neq f(t)$
  – homogeneous Markov Chain

Markov Process

- Accessibility
  - state S_i is accessible from state S_j if there is a sequence of transitions from S_j to S_i.
- Recurrent State
  - state S_i is called recurrent, if S_i can be returned to from any state which is accessible from S_i in one step, i.e. from all immediate neighbor states.
- Non Recurrent
  - if there exists at least one neighbor with no return path.

Markov Process

- sample chain



Which states
are recurrent
or non-recurrent?

Markov Process

- Classes of States
  - set of states (recurrent) s.t. any state in the class is reachable from any other state in the class.
  - note: 2 classes must be disjoint, since a common state would imply that states from both classes are accessible to each other.
- Absorbing State
  - a state S_i is absorbing iff

$$\sum_{j \neq i} \lambda_{ij} \Delta t = 0$$

Markov Process

- Irreducible Markov Chain
  - a Markov chain is called irreducible, if the entire system is one class
    - => there is no absorbing state
    - => there is no absorbing subgraph, i.e. there is no absorbing subset of states

Deriving Equations

- $P_i(t + \Delta t) =$ probability of being in state S_i after Δt

$$P_i(t + \Delta t) = P_i(t)[1 - \sum_{i \neq j} \lambda_{ij} \;\; \Delta t] + \sum_{i \neq j} P_j(t) \lambda_{ji} \;\; \Delta t$$

as $\Delta t \to 0$

(differentiate)

$$\lim_{\Delta t \to 0} \frac{P_i(t + \Delta t) - P_i(t)}{\Delta t} = -P_i(t) \sum_{i \neq j} \lambda_{ij} + \sum_{i \neq j} P_j(t) \lambda_{ji}$$

Deriving Equations

- With m states => m differential equations
- m-1 independent equations

$$\frac{dP_1(t)}{dt} = \sum_{j \neq 1} P_j(t)\lambda_{\text{j1}} \qquad - P_1(t)\sum_{j \neq 1} \lambda_{\text{1j}}$$

$$\frac{dP_i(t)}{dt} = \sum_{j \neq i} P_j(t)\lambda_{\text{ji}} \qquad - P_i(t)\sum_{j \neq i} \lambda_{\text{ij}}$$

$$\frac{dP_{m-1}(t)}{dt} = \sum_{j \neq m-1} P_j(t)\lambda_{\text{j(m-1)}} - P_{m-1}(t) \sum_{j \neq m-1} \lambda_{\text{(m-1)j}}$$

- m^{th} equation

$$1 = \sum_{\forall k} P_k(t)$$

Deriving Equations

- Matrix Notation

$$
\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \frac{dP_i(t)}{dt} \\ \frac{dP_{m-1}(t)}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sum_{j\neq 1}\lambda_{1j} & \lambda_{21} & \lambda_{31} & \cdots & & \lambda_{m1} \\ \lambda_{1i} & \lambda_{2i} & \cdots & -\sum_{j\neq i}\lambda_{ij} & & \lambda_{mi} \\ \lambda_{1(m-1)} & \cdots & & & -\sum_{j\neq m-1}\lambda_{(m-1)j} & \lambda_{m(m-1)} \\ 1 & 1 & & \cdots & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_i \\ P_{m-1} \\ P_m \end{bmatrix}
$$

Steady State Solutions

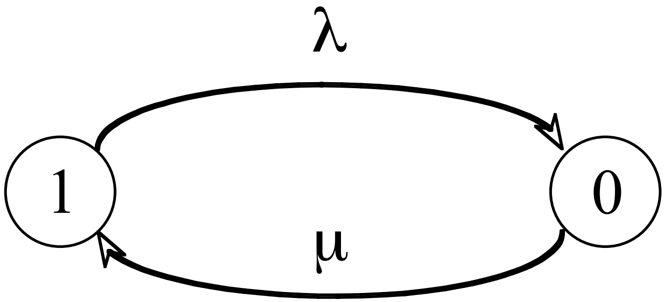
- Steady state solution:

$$\lim_{t \to \infty} \frac{dP_j(t)}{dt} = 0$$

- Steady state solution = Availability
  - set of linear alg. equations rather than linear differential equations

Steady State Solution

- Example: Simplex system with repair



λ = failure rate

μ = repair rate

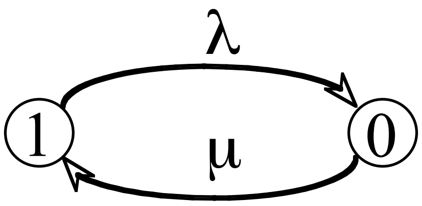
$$\begin{bmatrix} \frac{dP_0}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} -\mu & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

$$b = Ax$$

$$\begin{bmatrix} \dfrac{dP_0}{dt} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mu & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

Steady State Solution

- Simplex with Repair



- Solution:

$$P_0 = \frac{\lambda}{\mu + \lambda} \qquad P_1 = \frac{\mu}{\mu + \lambda}$$

- Steady State Availability

$$P_1 = \frac{\mu}{\mu + \lambda} = \lim_{t \to \infty} A(t)$$

- e.g.

$$\lambda = 10^{-3} \quad \Rightarrow \quad MTTF = 1000h$$

$$\mu = 10^{-1} \quad \Rightarrow \quad MTTR = 10h$$

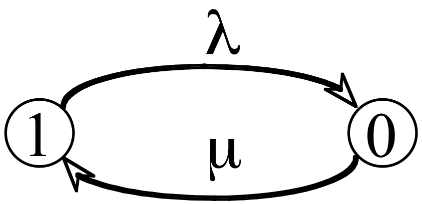
Availability:

The prob. that system is up

$$A = \frac{10^{-1}}{10^{-1} + 10^{-3}}$$

$$= 0.99 = 99\%$$

Transient Solution



- Simplex with Repair

$$\frac{dP_1(t)}{dt} = \mu P_0(t) - \lambda P_1(t)$$

with $P_0(t) + P_1(t) = 1$ we get

$$\frac{dP_1(t)}{dt} = \mu(1 - P_1(t)) - \lambda P_1(t)$$
$$= -P_1(t)(\mu + \lambda) + \mu$$

- $P_1'(t) + (\mu + \lambda) P_1(t) = \mu$ is a first order diff. equation

Transient Solution

- $P_1'(t) + (\mu + \lambda) P_1(t) = \mu$ has general solution

$$P_1(t) = \frac{\mu}{\mu + \lambda} + Ce^{-(\mu+\lambda)t}$$

- Get C by setting t=0

$$C = P_1(0) - \frac{\mu}{\mu + \lambda}$$

- Solution

$$P_1(t) = \frac{\mu}{\mu + \lambda} + \left(P_1(0) - \frac{\mu}{\mu + \lambda} \right) e^{-(\mu+\lambda)t}$$

Transient Solution

- with $t \longrightarrow \infty$ we get

$$P_1(t) = \frac{\mu}{\mu+\lambda} + \left(P_1(0) - \frac{\mu}{\mu+\lambda} \right) e^{-(\mu+\lambda)t}$$

$$= \frac{\mu}{\mu+\lambda}$$

$\longleftarrow$ our steady state solution (steady state availability)